

THE
Young Mathematician's Guide:
Being a PLAIN and EASY
INTRODUCTION
TO THE
MATHEMATICKS.

IN FIVE PARTS.

VIZ.

- I. **Arithmetick**, Vulgar and Decimal, with all the useful Rules;
And a General Method of Extracting the Roots of all Single Powers.
- II. **Algebra**, or Arithmetick in Species; wherein the Method of
Raising and Resolving Equations is rendered Easy; and illustrated
with Variety of Examples, and Numerical Questions. Also the whole
Business of Interest and Annuities, &c. performed by the Pen.
- III. The **Elements of Geometry**, contracted, and Analytically
demonstrated; with a new and easy Method of finding the Circle's
Periphery and Area to any assigned Exactness, by one Equation only:
Also a new way of making Sines and Tangents.
- IV. **Conick Sections**, wherein the chief Properties, &c. of the
Ellipsis, Parabola, and Hyperbola, are clearly demonstrated.
- V. The **Arithmetick of Infinites** explained, and rendered Easy;
with it's Application to superficial and solid Geometry.

With an APPENDIX of **Practical Gauging.**

By **JOHN WARD.**

The TENTH EDITION, carefully Corrected.

To which is added,
A SUPPLEMENT, containing the History of LOGARITHMS,
and an INDEX to the whole Work.

L O N D O N:

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To the HONOURABLE

Sir RICHARD GROSVENOR,
of *Eaton*, in the County *Palatine*
of *Chester*, Baronet.

S I R,

WHEN requested by some Booksellers in *London*, to revise and prepare this Treatise for a New Impression, and once resolved to answer their Demands; I was not long considering at whose Feet to lay it.

My Memory may indeed be impaired by Age, Misfortunes, and Accidents; nay, I am sensible it is so: But it must be entirely lost, when I am forgetful of the great Obligations I lie under to *Sir Richard Grosvenor*.

Your Hospitality and Generosity make you stand unenvied in the Abundance of Fortune. Any Upstart may contrive to spend a Great Estate; but it is a Felicity almost peculiar to Great Birth to become One.

Were I now to describe Liberality, without Profuseness; Steadiness in Principles, without any private View; Candour and Affability, Good Nature joined to sound Judgment, and a Serenity of Temper, which your Enemies will always find the Companion of true Courage; and then pronounce that you are possessed of all these good Qualities in as high a Degree as most Men living; No Gentleman that knows you well, would think I flattered you.

The *DEDICATION.*

Sir, Give me Leave to say, I honour your Character, and love your Person; My Expressions are uncourtly, my Stile unpolished, and therefore more proper to be prefixed to a Work wherein the Matters related are indeed clad in a plain and homely Dress; but they are true, and designed to propagate Mathematical Learning among such as desire to be introduced into that Sort of Knowledge; and I am extreamly pleased they are permitted to be sent into the World under your Protection.

That you may long live, to promote the Good of your Country, and that City in whose Interest you have so heartily engaged yourself; and that you may ever succeed in your own private Affairs, and live to enjoy all the Blessings that attend a quiet prudent Life, is the earnest Prayer of,

Honoured S I R,

Your most Obliged, Humble,

and Obedient Servant,

J. W A R D.

To

To the R E A D E R.

I Think it needless (and almost endless) to run over all the Usefulness, and Advantages of Mathematicks in General; and shall therefore only touch upon those two admirable Sciences, Arithmetick and Geometry; which are indeed the two grand Pillars (or rather the Foundations) upon which all other Parts of Mathematical Learning depend.

As to the Usefulness of Arithmetick, it is well known that no Business, Commerce, Trade, or Employment whatsoever, even from the Merchant to the Shop-keeper, &c. can be managed and carried on, without the Assistance of Numbers.

And as to the Usefulness of Geometry, it is as certain, that no curious Art, or Mechanick-Work, can either be invented, improved, or performed, without it's assisting Principles; though perhaps the Artist, or Workman, has but little (nay, scarce any) Knowledge in Geometry.

Then, as to the Advantages that arise from both these Noble Sciences, when duly joined together, to assist each other, and then apply'd to Practice, (according as Occasion requires) they will readily be granted by all who consider the vast Advantages that accrue to Mankind from the Business of Navigation only. As also from that of Surveying and Dividing of Lands betwixt Party and Party. Besides the great Pleasure and Use there is from Time-keepers, as Dials, Clocks, Watches, &c. All these, and a great many more very useful Arts, (too many to be enumerated here) wholly depend upon the aforesaid Sciences.

And therefore it is no Wonder, That in all Ages so many Ingenious and Learned Persons have employed themselves in writing upon the Subject of Mathematicks; but then most of those Authors seem to presuppose, that their Readers had made some Progress in that Sort of Learning before they attempted to peruse those Books, which are generally large Volumes, written in such abstruse Terms, that young Learners were really afraid of looking into those Studies.

These Considerations first put me (many Years ago) upon the Thoughts of endeavouring to compose such a plain and familiar Introduction to the Mathematicks, as might encourage those that were willing (to spend some Time that Way) to venture and proceed on with Cheerfulness; though perhaps they were wholly ignorant of it's first Rudiments. Therefore I began with their first Elements or Principles.

That

The P R E F A C E.

That is, I began with an Unit in Arithmetick, and a Point in Geometry; and from these Foundations proceeded gradually on, leading the young Learner Step by Step with all the Plainness I could, &c.

And for that Reason I published this Treatise (Anno 1707) by the Title of the Young Mathematician's Guide; which has answered the Title so well, that I believe I may truly say (without Vanity) this Treatise hath proved a very helpful Guide to near five thousand Persons; and perhaps most of them such as would never have looked into the Mathematicks at all but for it.

And not only so, but it hath been very well received amongst the Learned, and (I have been often told) so well approved on at the Universities, in England, Scotland, and Ireland, that it is ordered to be publickly read to their Pupils, &c.

The Title Page gives a short Account of the several Parts treated of, with the Corrections and Additions that are made to this Fifth Edition, which I shall not enlarge upon, but leave the Book to speak for itself; and if it be not able to give Satisfaction to the Reader, I am sure all I can say here in it's Behalf will never recommend it: But this may be truly said, That whoever reads it over, will find more in it than the Title doth promise, or perhaps he expects: it is true indeed, the Dress is but Plain and Homely, it being wholly intended to instruct, and not to amuse or puzzle the young Learner with hard Words, and obscure Terms: However, in this I shall always have the Satisfaction; That I have sincerely aimed at what is useful, tho' in one of the meanest ways; it is Honour enough for me to be accounted as one of the Under-Labourers in clearing the Ground a little, and removing some of the Rubbish that lay in the Way to this Sort of Knowledge. How well I have performed That, must be left to proper Judges.

To be brief; as I am not sensible of any Fundamental Error in this Treatise, so I will not pretend to say it is without Imperfections, (Humanum est errare) which I hope the Reader will excuse, and pass over with the like Candour and Good-Will that it was composed for his Use; by his real Well-wisher,

J. W A R D.

London, October 10th, 1706.

Corrected, &c. at Chester,
January 20th, 1722.

T H E

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A N

INTRODUCTION

TO THE

Mathematicks.

PART I.

PRÆCOGNITA.

THE Business of Mathematicks, in all it's Parts, both Theory and Practice, is only to search out and determine the true Quantity; either of Matter, Space, or Motion, according as Occasion requires.

By Quantity of Matter is here meant the Magnitude or Bigness of any visible thing, whose Length, Breadth, and Thickness, may either be measured, or estimated.

By Quantity of Space is meant the Distance of one thing from another.

And by Quantity of Motion is meant the Swiftnes of any thing moving from one Place to another.

The Consideration of these, according as they may be proposed, are the Subjects of the Mathematicks, but chiefly that of Matter.

Now the Consideration of Matter, with respect to it's Quantity, Form, and Position, which may either be Natural, Accidental, or Designed, will admit of infinite Varieties: But all the Varieties that are yet known, or indeed possible to be conceived, are wholly comprized under the due Consideration of these Two, Magnitude and Number, which are the proper Subjects of Geometry, Arithmetick, and Algebra. All other Parts of the Mathematicks being only the Branches of these three Sciences, or rather their Application to particular Cases.

Geometry is a Science by which we search out, and come to know, either the whole Magnitude, or some Part of any proposed Quantity; and is to be obtained by comparing it with another known Quantity of the same Kind, which will always be one of these, viz. **A Line** (or Length only); **A Surface** (that is, Length and Breadth); or a **Solid** (which hath Length, Breadth, and Depth, or Thickness); Nature admitting of no other Dimensions but these Three.

Arithmetick is a Science by which we come to know what Number of Quantities there are (either real or imaginary) of any Kind, contained in another Quantity of the same Kind: Now this Consideration is very different from that of Geometry, which is only to find out true and proper Answers to all such Questions as demand, how Long, how Broad, how Big, &c. But when we consider either more Quantities than one, or how often one Quantity is contained in another, then we have recourse to Arithmetick, which is to find out true and proper Answers to all such Questions as demand, how Many, what Number or Multitude of Quantities there are. To be brief, the Subject of Geometry is that of Quantity, with respect to it's Magnitude only; and the Subject of Arithmetick is Quantities with respect to their Number only.

Algebra is a Science by which the most abstruse or difficult Problems, either in Arithmetick or Geometry, are Resolved and Demonstrated; that is, it equally interferes with them both; and therefore it is promiscuously named, being sometimes called Specious Arithmetick, as by Harriot, Vieta, and Dr Wallis, &c. And sometimes it is called Modern Geometry, particularly the ingenious and great Mathematician Dr Edmund Halley, Savilian Professor of Geometry in the University of Oxford, and Royal Astronomer at Greenwich, giving this following Instance of the Excellence of our Modern Algebra, writes thus:

‘ The Excellence of the Modern Geometry (saith he) is in
 ‘ nothing more evident, than in those full and Adequate Solutions
 ‘ it gives to Problems; representing all the possible Cases at one
 ‘ View, and in one general Theorem many times comprehending
 ‘ whole Sciences; which deduced at length into Propositions, and
 ‘ demonstrated after the Manner of the Ancients, might well be-
 ‘ come the Subjects of large Treatises: For whatsoever Theorem
 ‘ solves the most complicated Problem of the Kind, does with a
 ‘ due Reduction reach all the subordinate Cases.’ Of which he
 gives a notable Instance in the Doctrine of Dioptricks for finding
 the Foci of Optick Glasses universally. (*Vide Philosophical Trans-*
actions, Numb. 205.)

Thus

Thus you have a short and general Account of the proper Subjects of those three noble and useful Sciences, Arithmetick, Geometry, and Algebra. I shall now proceed to give a particular Account of each; and first of Arithmetick, which is the Basis or Foundation of all Arts, both Mathematick and Mechanick; and therefore it ought to be well understood before the rest are meddled withal.

C H A P. I.

Concerning the several Parts of Arithmetick, with the Definition of such Characters as are used in this Treatise.

Arithmetick, or the Art of Numbering, is fitly divided into three distinct Parts, two of which are properly called *Natural*, and the third *Artificial*.

The first, being the most plain and easy, is commonly called *Vulgar Arithmetick* in whole Numbers; because every Unit or Integer concerned in it, represents one whole Quantity of some Species or thing proposed.

The second is that which supposes an Unit (and consequently the Quantity or thing represented by that Unit) to be Broken or Divided into equal Parts (either even or uneven) and considers of them either as pure Parts, viz. Each less than an Unit, or else of Parts and Integers intermixt. And is usually called the *Doctrine of Vulgar Fractions*.

The third, or Artificial Part, is called *Decimal Arithmetick*; being an Artificial Invention of managing Fractions or Broken Numbers, by a much more commodious and easy Way than that of *Vulgar Fractions*: For the several Operations performed in Decimals, differ but little from those in Whole Numbers: and therefore it is now become of general Use, especially in Geometrical Computations.

Arithmetick (in all it's Parts) is performed by the various ordering and disposing of Ten Arabick Characters or Numeral Figures (which by some are called Digits).

viz. $\left\{ \begin{array}{cccccccccc} \text{One, Two, Three, Four, Five, Six, Seven, Eight, Nine, Cypher,} \\ 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 0 \end{array} \right.$

The Use of these Characters is said to be first introduced into England near six hundred Years ago, viz., about the Year 1130, vide Dr Wallis's Algebra, Page 12.

The first of these *Characters* is called *Unity*, and represents one of any Kind of *Species* or *Quantity*. As one *World*, one *Star*, one *Man*, &c.

Viz. *Unity* is that by which every thing that is, is called one, (*Euclid*. 7. *Def*. 1.) and is the beginning of all *Numbers*. That is to say, *Number* is a *Multitude* of *Units*. *Euclid* 7. *Def*. 2.

For, one more one, makes *Two*; and one, more one, more one makes *Three*, &c. Which is the first and chief *Postulate*, or rather *Axiom* to *Arithmetick*.

Viz. { That $1+1=2$. $1+1+1=3$. $1+1+1+1=4$.
 $1+1+1+1+1=5$. And so on to 9.

Nine of these *Figures* were thus composed of *Units*, and differently form'd to represent so many *Units* put together into one *Sum*, as was intended each should denote: *Nine* being the greatest *Number* of *Units* that was then thought convenient to be expressed by one single *Character*; the last of the *Ten* is only a *Cypher*, or (as some phrase it) a *Nothing*, because of itself it signifies nothing; for if never so many *Cyphers* be Added to, or Subtracted from, any *Number*, they can neither increase nor diminish that *Number*; but yet, as a *Cypher* (or *Cyphers*) may be placed, the other *Figures* will become of different *Values* from what they were before, as will appear further on.

For the more convenient ordering of the afore said *Numeral Figures*, according to the several Varieties that happen in *Computations*; I do advise the young Learner to acquaint himself with the Signification of the following *Algebraick Signs* or *Characters*, which he will find of excellent Use, as being a much shorter, better, and more significant Way of denoting what is to be done (in most Operations) than can otherwise be expressed in Words at length.

Significations.

Signs Names.

+ } } Plus or
 } } more.

The Sign of *Addition*; as $8+7$ is 8 more 7, and signifies that the *Numbers* 8 and 7 are to be added into one *Sum*. The like is to be understood when several *Numbers* are connected together with the Sign +.

As $34+22+9+45$, &c. denotes these are all to be added into one *Sum*.

The

$-$ } { *Minus* } The Sign of *Subtraction*; as $9-6$ is 9 less
 } { *or less.* } 6, and signifies that 6 is to be taken from 9,
 that so their Difference may be found.

$\times$ } { *Into or* } The Sign of *Multiplication*; as 9×6 , is 9 in-
 } { *with.* } to 6, and signifies that 9 is to be Multiplied
 into or with 6.

$\div$ } { *By.* } The Sign of *Division*; as $8 \div 2$, is 8 by 2,
 and signifies that 8 is to be Divided by 2, also
 thus 2) 8 (4 or thus $\frac{8}{2}$ each signifying the same
 thing, to wit, 8 Divided by 2.

$=$ } { *Equal.* } The Sign of *Equality* or *Equation*, viz. when-
 ever this Sign $=$ is placed betwixt *Numbers* (or
 Quantities) it denotes them to be Equal, as
 $9=9$, or $9+6=15$, or $9-6=3$, &c. That
 is, 9 is Equal to 9, or 9 more 6 is Equal to 15,
 and 9 less 6 is Equal to 3, &c.

$::$ } { *So is.* } The Sign of *Proportion*, or that commonly
 called the *Golden Rule*, or *Rule of Three*, and
 $::$ is always placed betwixt the Two middle
 Terms or *Numbers* in Proportion. Thus
 $2:8::6:24$. To be read thus; as 2, is to 8;
 so is 6, to 24.

These *Signs* and their *Significations*, being perfectly learnt,
 will help to shorten the Work.

CHAP. II.

*Concerning the Principal Rules in Arithmetick, and how
 they are performed in Whole Numbers.*

THE Rules by which Numerical Operations are perform'd
 in all the Parts of *Arithmetick*, are many and various,
 several of them being form'd and rais'd as Occasion requires,
 when applied to *Practice*; yet they are all comprehended within
 the due Consideration of these Six, viz. *Numeration* (or *Pota-
 tion*)

tion) Addition, Subtraction, Multiplication, Division, and Evolution, or *Extraction of Roots*.

Sect. I. Of Numeration or Notation.

Numeration or *Notation*, teacheth to Read or Express the true Value of any *Number* when writ down; and consequently to write down any proposed *Number* according to it's true Value when it is named: And this consisteth of Two Parts.

1. The due Order of placing down Figures.
 2. The true valuing of each Figure in it's Place.
- Both which are plainly exhibited in the following Table.

Et.	Hundreds of Thousands of Millions,	And is the first Place in Counting.
	Tens of Thousands of Millions	
	Thousands of Millions	
	Hundreds of Millions	
	Tens of Millions	
	Millions	
	Hundreds of Thousands	
	Tens of Thousands	
	Thousands	
	Hundreds	
	Tens	
	Units	
	6 7 8 9 8 7 6 5 4 3 2 1	
Period of Thousands of Millions.		Period of Millions.
Period of Thousands.		Period of Units.

By this **Numeration Table** it is apparent, that the Order of Places is reckoned from the Right-hand towards the Left; the first Place of any *Number* being always that which is the outmost *Figure* to the Right-hand: and whatever *Figure* stands in that Place, doth only signify it's own simple Value, viz. so many *Units* as that *Figure* represents.

The second Place is that of *Tens*, and any *Figure* standing in that Place signifieth so many *Tens* as that *Figure* represents *Units*.

The

The third Place is *Hundreds*, the fourth Place *Thousands*, &c. That is, each Place towards the Left-hand is *Ten* Times the Value of that next it, towards the Right.

For Instance, suppose 759 were proposed to be read or pronounced according to the Value of each *Figure* as they now stand. The first *Figure* in this *Sum* is 9, because it stands in the Place of *Units*, and therefore signifies but it's own simple Value, to wit, 9 *Units*, or 9. The second *Figure* 5 stands in the Place of *Tens*, and therefore signifies Five *Tens* or *Fifty*. The *Figure* 7 stands in the third Place, or Place of *Hundreds*, and therefore it signifies *Seven Hundred*; and the whole *Sum* is to be read or pronounced thus, *Seven Hundred Fifty Nine*.

Note, Although the *Figure* 7 stands in the third Place (according to the Order of *Numbering*) yet when the whole *Sum* comes to be read, it is first pronounced; the reading of *Numbers* being performed like that of Letters or Words, always beginning with the outmost *Figure* towards the Left-hand, and so many *Figures* as are placed together without any Point, Comma, Line, or other Note of Distinction between them, are all but one *Sum*, and must be read as such.

For Example, 763596 is but one entire *Sum* or *Number*, notwithstanding it consists of six Places of *Figures*, and is thus read; *Seven Hundred Sixty Three Thousand, Five Hundred Ninety Six*.

The like is to be observed in reading or expressing the true Value of any *Sum* or *Rank* of *Numbers* consisting of *Seven*, *Eight*, *Nine*, or more Places of *Figures*, each *Figure* being to be valued according to it's Distance from the Place of *Unity*: As in the foregoing Table.

Now such Values may as well arise by *Cyphers*, as by other *Figures*; for Instance, 6 standing by itself, represents but Six *Units*: But if a *Cypher* be annexed to it thus, 60, then it becomes *Sixty*; for the *Cypher* possessing the Place of *Units*, hath thereby removed the 6 into the Place of *Tens*; and another *Cypher* more would make it 600, *Six Hundred*, &c.

Whence it may be noted, that although a *Cypher* of itself signify nothing (as hath been said before) yet being placed on the Right-hand of any *Figure*, it augments the Value of that *Figure* by advancing it into a higher Place than otherwise it would have been, had not the *Cypher* been there.

Take one Example more in *Numeration* (if you please, that in the Table) viz. 678987654321, which is, according as is there signified.

Six

*Six Hundred Seventy Eight Thousand Millions,
 Nine Hundred Eighty Seven Millions,
 Six Hundred Fifty Four Thousand,
 Three Hundred Twenty One Units.* Of any proposed *Species*
 or *Quantities* whatsoever.

And here it may be observed, that every third *Figure* from the Place of *Units*, bears the Name of *Hundreds*; which shews that if any great *Sum* be parted, or rather distinguished into Periods, of Three *Figures* in each *Period* (as in the foregoing Table), it will be of good Use to help the young Learner in the easier valuing and expressing that *Sum*.

Sect. 2. Of Addition.

Postulate or Petition.

That any given Number may be increased or made more, by putting another Number to it.

Addition is that *Rule* by which several *Numbers* are collected and put together, that so their *Sum* or *Total* Amount may be known.

In this *Rule* Two things being carefully observed, the Work will be easily performed.

1. The first is the true placing of the *Numbers*, so as that each *Figure* may stand directly underneath those *Figures* of the same Value, viz. place *Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c.

Then underneath the lowest Rank (always) draw a Line to separate the given *Numbers* from their *Sum* when it is found.

Example. If these *Numbers* 54327, and 2651, were given to be added together, they must be placed

$$\text{Thus, } \begin{array}{r} 54327 \\ 2651 \\ \hline \end{array}$$

2. The second thing to be observed is the due Collecting or Adding together each Row of *Figures* that stand over one another of the same Value: And that is thus performed.

Rule.

Always begin your Addition at the Place of Units, and Add together all the Figures that stand in that Place, and if their Sum be under Ten, set it down below the Line underneath it's own Place; but if their Sum be more than Ten, you must set down only the overplus, or odd Figure above the Ten (or Tens) and so many Tens as the Sum of those Units amount to, you must carry
 to

to the place of *Tens*; Adding them and all the Figures that stand in the place of *Tens* together, in the same manner as those of the *Units* were added; then proceed in the same order to the place of *Hundreds*, and so on to each place until all is done.

The Sum arising from those Additions will be the *Total* Amount required.

EXAMPLE 1.

Let it be required to find the Sum of the aforesaid Numbers,
viz. $\left\{ \begin{array}{r} 54327 \\ 2651 \end{array} \right.$

56978 the Sum required.

Beginning at the place of *Units*, I say 1 and 7 is 8, which being less than 10, I set it down (according to the *Rule*) underneath its own place of *Units*; and then proceed to the place of *Tens*, saying 5 and 2 is 7, which being less than 10, I set it down underneath its own place of *Tens*, and proceed to do the like at the place of *Hundreds*, and then at *Thousands*, setting each of their Sums underneath their own respective places: Lastly, because there is not any *Figure* in the lower Rank to be added to the Figure 5, which stands in the place of *Ten Thousands*, in the upper Rank, I therefore bring down the said 5 to the rest, placing it underneath its own place, and then I find that $54327 + 2651 = 56978$, the true Sum required.

EXAMPLE 2.

Suppose it were required to find the Sum of these Numbers, $3578 + 496 + 742 + 184 + 95$. These being placed, as before directed, will stand as in the Margin. Then beginning (as before) at the place of *Units*, say 5 and 4 is 9, and 2 is 11, and 6 is 17, and 8 is 25; set down the 5 *Units* underneath its own place of *Units*, and carry the 20, or two *Tens*, to the place of *Tens* (at which place they are only 2) saying, 2 and 9 is 11, and 8 is 19, and 4 is 23, and 9 is 32, and 7 is 39; set down the 9 underneath its own place of *Tens*, and carry the 30, or three *Tens* (which indeed is 300) to the place of *Hundreds*, at which place they are but 3, saying, 3 I carry and 1 is 4, and 7 is 11, and 4 is 15, and 5 is 20; here because there is no *Figure* overplus (as before) I set down a *Cypher* underneath the place of *Hundreds*, and carry the 2 *Tens* (or rather the 2000) to the place of *Thousands*, saying

C

(as

(as before) 2 I carry and 3 is 5, which being the last, I set it down underneath its own place, and all is finished. And find the *Sum* or *Total* amount to be $5095 = 3578 + 496 + 742 + 184 + 95$.

If this Example be well considered, it will be sufficient to shew the usual Method of *Addition* in whole *Numbers*; but to make all plain and clear, I shall shew the young Learner the Reason of carrying the *Tens* from one Degree or Row of *Figures*, to the next Superior Degree, which is done purely to save *Trouble*, and prevent the using of more *Figures* than are really necessary, as will appear by the following Method of adding together the same *Numbers* of the last Example.

Thus, add together each single Row of *Figures* by itself; as if there were no more but that one Row, setting down the *Sum* underneath its own place.

$$\left. \begin{array}{r} 3578 \\ 496 \\ 742 \\ 184 \\ \hline \end{array} \right\}$$

The *Sum* of the Row of *Units*, is
 The *Sum* of the Row of *Tens*, is
 The *Sum* of the Row of *Hund.* is
 The three *Thousand* brought down

$$\left. \begin{array}{r} 25 \\ 370 \\ 1700 \\ 3000 \\ \hline \end{array} \right\} \text{Add}$$

The *Sum* or *Total* Amount as before, is 5095

From hence I presume it will be easy to conceive the true Reason of carrying the aforesaid *Tens*; and also that *Cyphers* do not augment or increase the *Sum* in *Addition*. (See Page 4.)

I might have here inserted a Lineal Demonstration of this *Rule* of *Addition*; but I thought it would rather puzzle than improve a young Learner, especially in this place; besides the Reason of it is sufficiently evident from that Natural Truth of *the Whole being Equal to all its Parts taken together*. Euclid 1. Axiom 19.

That is, the *Numbers* which are proposed to be added together, are by that *Axiom* understood to be the several Parts, and their *Sum* or *Total* Amount found by *Addition* is understood to be the Whole.

And from thence is deduced the Method of proving the Truth of any *Operation* in *Addition*, viz. By parting or separating the given *Numbers* into Two Parcels (or more, according to the Largeness of it) and then adding up each Parcel by itself: For if those particular *Sums* so found, be added into one *Sum*, and that *Sum* prove Equal or the same with the *Total Sum* first found,

found, then all is right ; if not, care must be taken to discover and correct the Error.

E X A M P L E.

$$\begin{array}{r}
 \left. \begin{array}{r} 5647 \\ 3289 \\ 4016 \end{array} \right\} \text{The Sum of these Parts is, } 12952 \\
 \text{Add } \left\{ \begin{array}{r} \hline 2900 \\ 5007 \\ 1606 \end{array} \right\} \text{The Sum of these, is } \underline{9513} \\
 \hline
 \end{array}$$

The *Total Sum* of
all these Parts } 22465

The *Sum* of each }
Parcel put together } 22465

Sect. 3. Of Subtraction.

Postulate or Petition.

That any Number may be diminished, or made less, by taking another Number from it.

Subtraction is that *Rule* by which one *Number* is *deducted* or taken out of another, that so the *Remainder*, *Difference*, or *Excess* may be known.

As 6 taken out of 9, there *remains* 3. This 3 is also the *Difference* betwixt 6 and 9, or it is the *Excess* of 9 above 6.

Therefore the *Number* (or *Sum*) out of which *Subtraction* is required to be made, must be greater than (or at least equal to) the *Subtrahend* or *Number* to be *subtracted*.

Note, *This Rule is the Converse or Direct contrary to Addition.*

And here the same Caution that was given in *Addition*, of placing *Figures* directly under those of the same Value, viz. *Units* under *Units*, *Tens* under *Tens*, and *Hundreds* under *Hundreds*, &c. must be carefully observed ; also underneath the lowest Rank there must be drawn a Line (as before in *Addition*) to separate the given *Numbers* from their *Difference* when it is found.

Then having placed the lesser *Number* under the greater, the *Operation* may be thus performed.

R U I. E.

Begin at the Right Hand Figure or place of Units (as in Addition) and take or subtract the lower Figure in that place
C 2 from

from the Figure that stands over it, setting down the Remainder or Difference underneath its own place. If the Two Figures chance to be Equal, set down a Cypher: But if the upper Figure be less than the lower Figure, then you must add 10 to the upper Figure, or mentally call it 10 more than it is, and from that Sum subtract the lower Figure, setting down the Remainder (as before directed). Now because the 10 thus added, was suppos'd to be borrowed from the next superior place (viz. of Tens) in the upper Figures, therefore you must either call the upper Figure in that place from whence the 10 was borrowed, one less than really it is, or else (which is all one, and most usual) you must call the lower Figure in that place one more than it really is, and then proceed to Substraction in that place, as in the former; and so gradually on from one Row of Figures to another until all be done.

EXAMPLE 1.

Let it be required to find the *Difference* between 6785, and 4572. That is, let 4572 be *subtracted* from 6785.

These Numbers being placed down, as before directed, will stand

$$\begin{array}{r} \text{Thus } \left\{ \begin{array}{r} 6785 \\ 4572 \end{array} \right. \\ \hline 2213 \end{array}$$

Beginning at the place of *Units*, take 2 from 5 and there will remain 3 which must be set down underneath its own place, and then proceed to the place of *Tens*, taking 7 from 8, and there will remain 1, to be set down underneath its own place; again, at the place of *Hundreds*, take 5 from 7, and there remains 2, which set down, as before; lastly, take 4 from 6 and there will remain 2, which being set down underneath its own place, the Work is finished, and the *Difference* so found will be $2213 = 6785 - 4572$, as was required.

EXAMPLE 2.

The *Difference* between 5849, and 7496 is required.

Having placed the Numbers as in the Margin, begin at the place of *Units* (as before) and say 9 from 6 cannot be, but 9 from 16 and there remains 7, to be set down under its own place; next proceed to the place of *Tens*, where you must now pay the 10 that was borrowed to make the 6, 16, by counting the upper Figure 9 in that place one less than it is, saying 4 from 8 and there remains 4, or else (which is the most practised) say 1 I borrowed and 4 is 5 from

$$\begin{array}{r} 7496 \\ 5849 \\ \hline 1647 \end{array}$$

from 9 and there *remains* 4, to be set down under its own place (as before); again, at the place of *Hundreds*, say 8 from 4 that cannot be, but 8 from 14 there will *remain* 6 to be set down; and here I have borrowed 10 (as before) which must be paid in the same manner as the other 10 was, *viz.* either by calling the 7 in the upper Rank but 6, saying 5 from 6 there *remains* 1, or else by saying 1 borrowed and 5 is 6 from 7 and there *remains* 1, which being set down under its own place all is done, and the *Difference* required will be $1647 = 7496 - 5849$.

EXAMPLE 3.

From 830476
Take 741068

Remains 89408

By this Example you may perceive that *Cyphers* in the *Subtrahend*, *viz.* in the *Numbers* to be *subtracted*, do not diminish the *Number* from whence *Substraction* is made. See Page 4.

These Three Examples, I presume, may be sufficient to shew the young Learner the Method of *Subtracting* whole *Numbers*; as for the Reason thereof it is the same with that of *Addition*, Page 10, *viz.* of the *Whole* being *Equal* to all its *Parts* taken together.

That is, in this *Rule* the *Number* from which *Substraction* is required to be made, is understood to be the *Whole*, and the *Subtrahend* or *Number* to be *subtracted*, is supposed to be a part of that *Whole*; consequently if that *Part* be taken from the *Whole*, the *Remainder* will be the other part.

From hence is deduced the common Method of proving *Substraction*, by adding together the *Subtrahend* and the *Remainder*. For if the *Sum* of those Two (which are here called *Parts*) be equal to the *Number* from whence *Substraction* was made (which is here called the *Whole*) then the *Work* is right; if not, care must be taken to discover and correct the *Error*.

EXAMPLE.

From 59435
Take 47608
----- } Add

Proof { 11827
 { 59435

The *Sum* which is equal to the *Number* from whence *Substraction* was made.

Or

Or from the abovesaid Reason, it will be easy to conceive how to prove the Truth of *Subtraction* by *Subtraction*.

For if from 59435 being here the whole,
there be taken 47608 as part of that whole ;
there will remain 11827 the other part (as before)
And if from 59435 the whole, there be *subtracted* the
last part, viz. 11827
there will remain 47608 the first part, or *Number* which was
required to be first *Subtracted*.

From 75643
Take 9000
Remains 66643

From 7000000
Take 986432
Remains 6013568

Sect. 4. Of Multiplication.

Multiplication is a *Rule* by which any given *Number* may be speedily increased, according to any proposed *Number* of *Times*.

That is, One *Number* is said to Multiply another, when the *Number* multiplied is so often added to itself, as there are *Units* in the *Number* multiplying; and another *Number* is produced. (Euclid 7. Def. 15.)

To perform *Multiplication*, there is required two given *Numbers*, called *Factors*.

The First is the *Number* to be multiplied, which is generally put the greater of the Two *Numbers*, and is commonly called the *Multiplicand*.

The other is that *Number* by which the First is to be multiplied, and is usually called the *Multiplicator* or *Multiplier*; and this denotes the *Number* of *Times* that the *Multiplicand* is required to be added to itself. For so many *Units* as are contained in the *Multiplier*, so many times will the *Multiplicand* be really added to itself (as per Euclid above). And from thence will arise a Third *Number*, called the *Product*. But in Geometrical Operations it is called the *Rectangle* or *Plain*.

For instance; suppose it were required to increase 6 four times, that is, to multiply 6 into or with 4. These two *Numbers* are to be set (or placed) down as in *Addition* or *Subtraction*.

Thus

Thus $\left\{ \begin{array}{l} 6 \text{ Multiplicand,} \\ 4 \text{ Multiplier,} \end{array} \right\}$ or Factors.

Product 24 viz. 4 times 6 is 24, as plainly appears by *Addition*, viz. By setting down 6 four times, and then adding them together into one Sum,

$$\begin{array}{r|l} 1 & 6 \\ 2 & 6 \\ 3 & 6 \\ 4 & 6 \\ \hline & 24 \end{array} \left. \vphantom{\begin{array}{r|l} 1 & 6 \\ 2 & 6 \\ 3 & 6 \\ 4 & 6 \end{array}} \right\} \text{Add}$$

From hence it is evident, that Multiplication is only a Concise or Compendious Way of adding any given Number to itself, so often as any Number of Times may be proposed.

Before any Operation can be readily performed in *Multiplication*, the several *Products* of the single *Figures* one into another must be perfectly learn'd by Heart, viz. That 2 times 2 is 4, that 3 times 3 is 9, and 3 times 6 is 18, &c. According as they are expressed in the following *Table*; wherein I have omitted multiplying with 2, it being so very easy that any one may do it.

Multiplication Table.

3×3=9	4×4=16	5×5=25	6×6=36	7×7=49	8×8=64
3×4=12	4×5=20	5×6=30	6×7=42	7×8=56	8×9=72
3×5=15	4×6=24	5×7=35	6×8=48	7×9=63	9×9=81
3×6=18	4×7=28	5×8=40	6×9=54		
3×7=21	4×8=32	5×9=45			
3×8=24	4×9=36				
3×9=27					

I think it needless to give any *Explanation* of this *Table*; for if the *Signs* and their *Significations* be well understood, (*vide* page 5) it must needs be easy. Only this may be noted, that $4 \times 3 = 3 \times 4$, or $7 \times 5 = 5 \times 7$, &c.

That is, 3 times 4 is the same with 4 times 3, or 5 times 7 is the same with 7 times 5, &c. The like must be understood of all the rest in the *Table*.

And when all these single *Products* are so perfectly learn'd by Heart, as to be said without pausing; you may then proceed (but not 'till then) to the Business of *Multiplication*; which will be found very easy, if the following *Rule* (and *Examples*) be carefully observed.

R U L E.

Always begin with that Figure which stands in the Units place with the Multiplier, and with it multiply the Figure which stands

in

in the *Units* place of the *Multiplicand*; if their *Product* be less than *Ten*, set it down underneath its own place of *Units*, and proceed to the next *Figure* of the *Multiplicand*. But if their *Product* be above *Ten* (or *Tens*) then set down the *Overplus* only (or odd *Figure*, as in *Addition*) and bear (or carry) the said *Ten* or *Tens* in mind until you have multiplied the next *Figure* of the *Multiplicand*, with the same *Figure* of the *Multiplier*; then to their *Product* add the *Ten* or *Tens* carried in mind, setting down the *Overplus* of their *Sum* above the *Tens*, as before: and so proceed on in the very same manner, until all the *Figures* of the *Multiplicand* are multiplied with that *Figure* of the *Multiplier*.

EXAMPLE 1.

Suppose it were required to multiply 3213 into or with 3.

$$\begin{array}{r} 3213 \text{ Multiplicand } \\ 3 \text{ Multiplier, } \end{array} \left. \vphantom{\begin{array}{r} 3213 \\ 3 \end{array}} \right\} \text{or Factors.}$$

Product 9639

Beginning at the *Units* place, say 3 times 3 is 9, which, because it is less than *Ten*, set down underneath its own place; and proceed to the next place of *Tens*, saying 3 times 1 is 3, which set down underneath its own place; then to the next place, viz. of *Hundreds*, saying 3 times 2 is 6, which set down, as before; lastly, at the place of *Thousands*, say 3 times 3 is 9, which being set down underneath its own place, the Operation is finish'd; and the true *Product* is $9639 = 3213 \times 3$, as was required.

EXAMPLE 2.

Let it be required to multiply 8569 into 8. Set down these *Numbers* as before,

$$\begin{array}{r} \text{Thus } \left\{ \begin{array}{r} 8569 \\ 8 \end{array} \right. \\ \hline 68552 \end{array}$$

Beginning at the *Units* place, say 8 times 9 is 72, set down the 2 underneath its own place of *Units*, and bear the 70, or 7 *Tens* in mind, and proceed to the next *Figure* of the *Multiplicand* (at which place the 7 *Tens* are only 7) saying 8 times 6 is 48, and the 7 carried in mind is 55; set down the odd 5 underneath its own place of *Tens*, and carry the 50 (which is really 500) to the next place (viz. of *Hundreds*) at which place it is only 5, where say, 8 times 5 is 40, and the 5 carried in mind is 45; set down the 5 underneath its own place, and carry the 40 or 4 *Tens* (which is really 4000) to the next

next place, viz. of *Thousands*, saying, 8 times 8 is 64, and 4 carried in mind is 68. (Now this being the last Place or *Figure* to be multiplied) Set down the whole *Product* 68, and the Work is done.

So that, $8569 \times 8 = 68552$, the *Product* required.

Now the Reason of this and all other the like Operations, may be easily conceived from this which follows.

8 5 6 9 } The same *Factors* as before.

7	2	{ Here 8 times 9 is 72, as before, because the 9 stands in the <i>Units</i> place. Now here it is not really 8 times 6 = 48, but it is 8 times 60 = 480, because the 6 stands in the place of <i>Tens</i> . And here it is not 8 times 5 = 40, but it is really 8 times 500 = 4000, because the 5 stands in the place of <i>Hundreds</i> . Lastly, because the 8 in the <i>Multiplicand</i> stands in the place of the <i>Thousands</i> , it is therefore 8 times 8000 = 64000, and not 8 times 8 = 64. The Sum of the particular <i>Products</i> , which gives the true <i>Product</i> , as before.
4	8	
4	0	
6	4	
6	8	
6 8 5 5 2		

By what hath been already said, with a little Consideration had to the *Examples*, I presume the Learner may easily understand how to multiply whole Numbers with any single *Figure*. And when it is required to multiply with more than one; then so many Figures as there are in the *Multiplier*, so many particular *Products* there must be.

That is, all the *Figures* of the *Multiplicand* must be multiplied with every single *Figure* of the *Multiplier*, as if there were but one single *Figure*: and the Sum of all those particular *Products*, will be the true *Product* required. But in those Operations, great Care must be taken in setting down the particular *Products* (which arise by each multiplying *Figure*) in their proper places. Which will be easily done, if the following Directions be carefully observed.

Viz. { Always place the first *Figure* (or *Cypher*) of every particular *Product*, directly underneath the multiplying *Figure*. Or thus:

The First *Figure* (or *Cypher*) of the second particular *Product* must stand directly under the second *Figure* (or place) of the First *Product*; and the First *Figure* (or *Cypher*) of the Third particular

particular Product, must stand directly underneath the Third Figure of the First Product: And so on until all is done.

Now the Reason of placing the first Figure of every particular Product in their Order, will be very obvious to any one that considers the last Example; wherein the Cyphers are only set down to shew the true Distance of the first Figure in each particular Product from the Units place. And altho' it is not usual to set down Cyphers in this manner; yet they are always supposed to be there: That is, their Places are always left void, as in the two following Examples; wherein I have placed Points instead of Cyphers.

EXAMPLE 3.

Let it be required to multiply 78094, into or with 7563.

78094	} Factors.	
7563		
234282	The First particular <i>Product</i> with	3
468564.	The Second particular <i>Product</i> with	60
390470..	The Third particular <i>Product</i> with	500
546658...	The Fourth particular <i>Product</i> with	7000
590624922	The <i>Total</i> , or true <i>Product</i> required.	

EXAMPLE 4.

Suppose it be required to multiply 57498 into 60008.

57498	
60008	
459984	The Product with 8
344988....	The Product with 6000
3450339984 = 57498 × 60008, as was required.	

Here you may observe, that I pass over the Cyphers, and only take care of placing the first Product of the last Figure, viz. of 60000 according to the foregoing Directions.

When there is a Cypher or Cyphers, to the Right-hand either of the Multiplicand or Multiplier, or to both; in that case multiply the Figures as before; neglecting the Cyphers until the particular Products are added together; Then to their Sum annex so many Cyphers as are in either or both the Factors. As in these:

EXAMPLE

EXAMPLE 5.

$$\begin{array}{r} 9538 \\ 4600 \\ \hline 57228 \\ 38152 \\ \hline 43874800 \end{array}$$

EXAMPLE 6.

$$\begin{array}{r} 87600 \\ 79 \\ \hline 7884 \\ 6132 \\ \hline 6920400 \end{array}$$

EXAMPLE 7.

$$\begin{array}{r} 785000 \\ 56900 \\ \hline 7065 \\ 4710 \\ 3925 \\ \hline 44666500000 \end{array}$$

Take a few Examples without their Work at large.

$$\begin{aligned} 75649 \times 579 &= 43800771 \\ 687000 \times 356 &= 244572000 \\ 530674 \times 45007 &= 23884044718 \\ 7901375 \times 30000 &= 237041250000 \\ 537084000 \times 590700 &= 317255518800000 \\ 102030405 \times 504030201 &= 51426405540261405 \\ 987654321 \times 123456789 &= 121932641112635269 \end{aligned}$$

Note, If it be required to multiply any Number with 10, 100, 1000, 10000, &c. it is only annexing the Cyphers of the Multiplier to the Figures of the Multiplicand, and the Work is done.

$$\text{Thus } \begin{cases} 578 \times 10 = 5780. & 578 \times 1000 = 578000 \\ 578 \times 100 = 57800. & 578 \times 10000 = 5780000, \&c. \end{cases}$$

These Examples (being well understood) are sufficient to instruct the Learner all the Varieties that can happen in multiplying of whole Numbers, according to the Method generally practised: However it may not be amiss to shew here how Multiplication may be performed (with many Figures) by Addition only.

EXAMPLE.

Let it be required to multiply 879654 into 79863.

In order to perform this (or any Operation of this kind) by Addition only; you must make a Tariffa or small Table of the given Multiplicand, in this manner:

First, Make a small Column, and in it place gradually downward the Nine single Figures; viz. 1, 2, 3, 4, 5, &c.

D 2

Then

Then against the *Figure 1*, set down the *Multiplicand* (which in this *Example* is 879654) and against the *Figure 2*, set down the double of the *Multiplicand*, found by adding it to itself; To this double add the *Multiplicand*, setting down their *Sum* against the *Figure 3*. And so proceed on by a continued *Addition* until there be Ten times the *Multiplicand* in the *Table*; which if the *Work* is true, will be the *Multiplicand* itself with a *Cypher* to the Right-hand of it, as in the annexed *Table*. This being done, it will be easy to conceive, that the *Figures* in the small Column of the *Table*, do respectively represent those of the *Multiplier*: And that the *Numbers* against any of those *Figures* in the small Column, will be the true *Product* of the *Multiplicand* agreeing to any *Figure* of the *Multiplier*; as plainly appears by the *Work* of this *Example*

1	879654
2	1759308
3	2638962
4	3518616
5	4398270
6	5277924
7	6157578
8	7037232
9	7916886
10	8796540

Then

879654 } The *Factors* as before.
79863 }

Against { 3, in the *Table* is 2638962 = 879654 × 3
6, is 5277924 = 879654 × 60
8, is 7037232 = 879654 × 800
9, is 7916886 = 879654 × 9000
7, is 6157578 = 879654 × 70000

The *Product* required 70251807402 = 879654 × 79863

Note, This Method of Tabulating the *Multiplicand*, is both easy and certain; being neither subject to Errors, nor burdensome to the Memory, and therefore in large Calculations it may be found very useful. But for common Practice the useful Method (as in *Page 18*, &c.) is best, and to be preferred before this.

Most *Masters* that teach (and several *Authors* that write of) *Arithmetick*, do teach to prove the Truth of *Multiplication*, by casting away all the *Nines* that are contained in both the *Factors*, and their *Product*; but because that Method is very erroneous, as might be easily shewed; I shall therefore omit inserting it, and leave the Proof of *Multiplication* to the next *Section*, where-in (I presume) the Reason and Proof, both of it, and *Division*, will plainly appear.

Sect.

Sect. 5. Of Division.

Division is a *Rule* by which one *Number* may be speedily *subtracted* from another, so many times as it is contained therein.

That is, It speedily discovers how often one *Number* is contained (or may be found) in another: And to perform that there are required *Two Numbers* to be given.

1. The one of them is that *Number* which is proposed to be *divided*, and is called the *Dividend*.

2. The other is that *Number* by which the said *Dividend* is to be *divided*, and is called the *Divisor*.

And by comparing these *Two*, viz. the *Dividend* and the *Divisor* together, there will arise a *Third Number*, called the *Quotient*; which shews how often the *Divisor* is contained in the *Dividend*, or into what *Number* of *Equal Parts* the *Dividend* is then divided. Therefore,

Division is by Euclid fitly termed the measuring of one Number by another, viz. one *Number* is said to measure another by that *Number*, which when it *multiplies*, or is *multiplied* by it, it produceth. *Euclid 7. Def. 23.*

And, if a *Number* measuring another, multiply that *Number* by which it measureth, or be multiplied by it, it produceth the *Number* which it measureth. *Euclid 7. Axiom 9.*

That is to say, If that *Number* which divides another (called the *Divisor*) be multiplied with the *Number* which is produced by *Division* (called the *Quotient*) their *Product* will be the *Number* divided or *Dividend*. Whence it follows, that *Division* and *Multiplication* are the *Converse* and *Direct Contrary* one to another (as *Subtraction* is to *Addition*) and do mutually prove the *Truth* of each other's *Operations*.

I shall therefore make choice of the foregoing *Examples* in *Multiplication*, in order (as I presume) to render the *Business* of *Division* more plain and easy.

First, Let it be required to find how often 6 is contained in 24. That is, to divide 24 by 6.

N. B. Always place down the given *Numbers* in this Order; First set down the *Divisor*, and to the *Right-hand* of it draw a crooked Line; then set down the *Dividend*, and to the *Right* of it draw another crooked Line, in which must be placed the *Quotient Figure*, or *Figures* as they become found.

Thus

Dividend.

Thus *Divisor* 6) 24 (4 the *Quotient*.

Here I consider how many times 6 there is in 24, and find it 4, viz. 4 times 6 is 24, therefore 4 is the true *Quotient* or *Answer* required.

This is apparent by *Subtraction*, as in the Margin; where 24 the *Dividend* is set down, and from it 6 the *Divisor* continually *subtracted* so often as it can be, which is just 4 times. Therefore 4 is the true *Quotient* or *Answer* required.

Compare this with the
Example, Page 15.

	24
1	6
	18
2	6
	12
3	6
	6
4	6
	0

Corollary.

From hence it is evident; that *Division* is but a *concise* or *compendious Method* of *subtracting* one *Number* from another, so often as it can be found therein; for if the *Divisor* be continually *subtracted* from the *Dividend*, accounting an *Unit* (or 1) for each time it is *subtracted* (as above) the *Sum* of those *Units* will be the *Quotient*.

All Operations in *Division* do begin contrary to those of *Multiplication*, viz. at the *First Figure* to the Left-hand, or that of the highest Value, and decrease the *Dividend* by a repeated *Subtraction* of each *Product* arising from the *Divisor* when multiplied into the *Quotient Figure*. And the only difficulty in *Division* of whole *Numbers* (or indeed of any *Numbers*) lies in making choice of such a *Quotient Figure*, as is neither too big, nor too little; and that may be easily obtained by observing the following *Rule*, which hath two *Cases*.

R U L E.

Case I. As often as the *First Figure* of the *Divisor* is taken from the *First Figure* of the *Dividend*: So often must the *Second Figure* of the *Divisor* be taken from the *Second Figure* of the *Dividend*, when it is joined with what *Remains* of the *First*. And as often must the *Third Figure* of the *Divisor* be taken from the *Third Figure* of the *Dividend*, &c.

But if the *First Figure* of the *Divisor* cannot be taken from the *First Figure* of the *Dividend*. Then;

Case

Case 2. So often as the First Figure of the Divisor is taken from the Two First Figures of the Dividend, so often must the Second Figure of the Divisor be taken from the Third Figure of the Dividend, when it is joined with what remained of the Second: And so often must the Third Figure of the Divisor be taken from the Fourth Figure of the Dividend, &c.

That is, the Quotient Figure must be such, as being multiplied into the Divisor, will produce a Product equal to such a part of the Dividend as is then taken for that Operation: But if such a Product cannot be exactly found, then the next less must be taken, and ordered, as in the following Examples: of which let that in Page 16 be the first, wherein there was given 8569 the Multiplicand, and 8 the Multiplier. To find the Product 68552. Let us here suppose the said Product 68552, and 8 the Multiplier, both given; thence to find the Multiplicand. That is, Let it be required to divide 68552 by 8.

Dividend
Divisor 8) 68552 (Quotient when found.

According to the Rule, Case 1. I compare 8 the Divisor with 6 the First Figure of the Dividend, and finding I cannot take it from that, I then consider (by Case 2.) how often 8 can be taken from 68, the two first Figures of the Dividend, and find it may be taken 8 times; for 8 times 8 is 64, being the greatest Product of 8 (into any Figure) that can be taken from 68. I therefore place 8 in the Quotient, and with it multiply 8 the Divisor, setting down their Product underneath the said Two First Figures of the Dividend, subtracting it from them, and then the Work will stand

Thus 8) 68552 (8
64
———
4

In order to a Second Operation, I make a Point under the next Figure of the Dividend, viz. under the 5, and bring it down underneath in its own place to the Remainder 4, which will by that means become 45. Then I consider how many times 8 can be taken from 45, and find it may be 5 times; for 5 times 8 is 40, I therefore place 5 in the Quotient, and with it multiply 8 the Divisor, setting down and subtracting their Product, as before. Then the Work will stand

Thus

Thus 8) 68552 (85
64.

45

40

5

For a Third Operation, I make a *Point* under the next *Figure* of the *Dividend*, viz. under the 5, and bring it down, as before, proceeding in all respects, as before; and then the Work will stand

Thus 8) 68552 (856
64..

45

40

55

48

7

Lastly, I point and bring down the 2, viz. the last *Figure* of the *Dividend* to the *Remainder* 7, which will then become 72, and proceeding as in the other *Operations*, I find that 8 the *Divisor* can be taken just 9 times from 72, and the Work is finished, and will stand

Thus 8) 68552 (8569
64...

45

40

55

48

72

72

(0)

The true *Quotient* is found to be 8569, being exactly the *Eighth* part of 68552, or the *Multiplicand* of the proposed *Example* of *Multiplication*. As was required.

The Reason of the *Operations* will be very plain to any one that will a little consider of it, as follows:

Divisor

Divisor 8) 6 8 5 5 2 (8000. The First Quotient Figure.

Subtract $\left| \begin{array}{c|c|c|c|c} 6 & 4 & 0 & 0 & 0 \end{array} \right|$ { This *Product* of the *Divisor* into the *Quotient* is 64000, viz. 8 times 8000; the *Quotient Figure* being always of the same *Value* or *Degree* with that *Figure* under which the *Unit's* place of its *Product* stands.

Divisor 8) $\left| \begin{array}{c|c|c|c} 4 & 5 & 5 & 2 \end{array} \right|$ (500. The Second Quotient Figure.
Subtract $\left| \begin{array}{c|c|c|c} 4 & 0 & 0 & 0 \end{array} \right|$ { And here the *Product* is 4000, viz. 8 times 500, not 8 times 5.

Divisor 8) $\left| \begin{array}{c|c|c} 5 & 5 & 2 \end{array} \right|$ (60. The Third Quotient Figure.
Subtract $\left| \begin{array}{c|c|c} 4 & 8 & 0 \end{array} \right|$ { Also here the *Product* is 480, viz. 8 times 60, for the *Reasons* above said.

Divisor 8) $\left| \begin{array}{c|c} 7 & 2 \end{array} \right|$ (9. The Fourth Quotient Figure.
Subtract $\left| \begin{array}{c|c} 7 & 2 \end{array} \right|$ { Now here the *Product* is but 72, viz. 9 times 8, because the 9 stands in the place of *Units*.

Remains (00) Now the *Sum* of all the several *Quotients*, viz. $8000 + 500 + 60 + 9 = 8569$, as before.

If the *Process* of this *Example* be well considered and compared with that of *Multiplication*, Page 17, it will evidently appear to be only the *Converse* of that; for the particular *Products* are alike in both, only that which is *last* there, is *first* here; there they are *added*, here they are *subtracted*. So that whoever understands the true *Reason* of the one, must needs understand the *Reason* of the other, and then *Division* will become very *easy*, although the *Divisor* consists of several places of *Figures*.

EXAMPLE.

Let it be required to divide 590624922 by 7563.

Dividend.

Divisor 7563) 590624922 (

'Tis plain at the first sight, that 7563 the *Divisor*, cannot be taken from 5906, the like *Number* of *Figures* in the *Dividend*.

Therefore, by the *Second Case* of the *Rule* (Page 23.) there must be allowed Five *Figures* of the *Dividend*, viz. 59062 for the *First Operation* or *Quotient*; that so the *First Figure* 7 of the *Divisor* may be taken out of the two *First Figures*, viz. 59 of the *Dividend*, &c.

E

Then

Then I proceed (*per Case 2.*) and consider how often 7 may be taken from 59, and find it may be taken 8 times, for 8 times 7 is but 56, which I mentally *subtract* from 59, and there *remains* 3; to this 3 I mentally adjoin the *Third Figure* of the *Dividend*, *viz.* 0, which makes it 30, out of which I must take the *Second Figure* of the *Divisor*, *viz.* 5, so often as I took the 7 from 59, which was 8 times: But that cannot be, for 8 times 5 is 40, which is more than 30, therefore 8 is too big a *Figure* to be placed in the *Quotient*; yet, hence I conclude, that the next less, *viz.* 7 may be taken without any further *Trial*. I therefore place 7 in the *Quotient*, and with it *multiply* the *Divisor*, setting down their *Product* under the *Dividend*, and *subtract* it from thence, as in the other *Example*, and then the *Work* will stand

$$\begin{array}{r} \text{Thus } 7563) 590624922 \text{ (7} \\ \underline{52941} \\ 6121 \end{array}$$

In order to a *Second Operation*, I make a *Point* under the next *Figure* of the *Dividend*, *viz.* under the 4, and bring it down to the *Remainder* 6121, which will then become 61214, with which I proceed in all respects as I did before with the 59062, and find the next *Quotient Figure* will be 8, with which I *multiply* the *Divisor*, &c. and *subtract* their *Product* from the said 61214. Then the *Work* will stand

$$\begin{array}{r} 7563) 590624922 \text{ (78} \\ \underline{52941} \\ 61214 \\ \underline{60504} \\ 710 \end{array}$$

To this *Remainder* 710, I point and bring down the next *Figure* of the *Dividend*, *viz.* 9, which makes it 7109; now because the *Divisor* 7563 cannot be taken from 7109, I therefore place a *Cypher* in the *Quotient*.

And this must always be carefully observed, *viz.* That for every *Figure* or *Cypher*, which is brought down from the *Dividend*, in order to a new *Operation*, there must always be either a *Figure* or *Cypher*, set down in the *Quotient*. Then the *Work* will stand

Thus

Thus 7563) 590624922 (780
52941..

61214
60504

7109

To this 7109, I bring down another *Figure* of the *Dividend*, viz. 2, and then it will become 71092; then I consider how often 7 can be taken from 71, &c. (just as at the first Operation,) and find it may be taken 9 times, therefore I set down 9 in the *Quotient*, and with it multiply the *Divisor*, setting down and subtracting their *Product*, as before; Then the Work will stand

Thus 7563) 590624922 (7809
52941...

61214
60504

71092
68067

3025

To this *Remainder* 3025, I point and bring down the last *Figure* 2 of the *Dividend*, which makes it 30252; then proceeding in all respects as before, I find the *Quotient Figure* to be 4, with it I multiply the *Divisor*, setting down and subtracting their *Product* as before, and then the Work will stand

Thus 7563) 590624922 (78094
52941....

61214
60504

71092
68067

30252
30252

(00000)

Here the Work is ended, and I find the *Quotient* to be 78094, being the true *Multiplicand* of the proposed *Example* of *Multiplication*, Page 18.

That is, 7563 is contained in 590624922 just 78094 times, &c.
E 2 If

If the Work of this *Example* be considered and compared with the *Rule* (Page 22.) the whole Business of *Division* will be easy; for indeed the only Difficulty (as I said before) lies in making choice of a true *Quotient Figure*, which cannot well be done according to the Common Method of *Division*, without Trials, yet those Trials need not be made with the whole *Divisor*, (as appears by this last *Example*) for by the two First Figures of the *Divisor* all the rest are generally regulated; except the Second Figure chance to be 2, 3, or 4, and at the same time the Third Figure be 7, 8, or 9, then indeed respect must be had to the Third Figure, according as the *Rule* directs.

However, if those Trials are thought too troublesome, they may be avoided, and the same *Quotient Figure* may both easily and certainly be found by help of such a small *Table* made of the *Divisor*, as was of the *Multiplicand* in Page 20.

EXAMPLE 4.

Let it be required to divide 70251807402 by 79863. See the *Example of Multiplication*, Page 20, and as there directed make a *Table of the Divisor* 79863,

Thus,

	<i>Divisor.</i>	<i>Dividend.</i>	<i>Quotient.</i>
1	79863)	70251807402	(879654
2	159726	638904.....	
3	239589	636140	
4	319452	559041	
5	399315	770997	
6	479178	718767	
7	559041		
8	638904	522304	
9	718767	479178	
10	798630	431260	
		399315	
		319452	
		310452	
		(000000)	

The Work of this Operation I presume may be easily understood. For those *Figures* in the *Table* are the *Product* of the *Divisor* into all the 9 *Figures*; consequently those *Figures* in the small Column do shew what *Figure* is to be placed in the *Quotient*; without any doubtful Trials of the *Divisor*, with the *Dividend*, as before.

This Method of Tabulating the *Divisor* may be of good Use to a Learner; especially until he is well practised in *Division*; yea, and even then if the *Divisor* be large, and a *Quotient* of many *Figures* be required; as in resolving of high *Equations*, and calculating of *Astronomical Tables*, or those of Interest, &c.

Hitherto

Hitherto I have made choice of *Examples* wherein the *Dividend* is truly measured or *divided* off by the *Divisor*, without leaving any *Remainder*, being exactly composed of the *Divisor* and *Quotient*. But it most usually falls out, that the *Divisor* will not exactly measure the *Dividend*; in which case the *Remainder* (after *Division* is ended) must be set over the *Divisor* with a small Line betwixt them adjoining to the *Quotient*.

EXAMPLE 5.

Suppose it were required to divide 379 by 5.

$$5) 379 \quad (75 \frac{4}{5} \text{ the Remainder, the Divisor.})$$

35.

29

25

Remains (4)

EXAMPLE 6.

Again, Let it be required to divide 43789 by 67.

$$67) 43789 \quad (653 \frac{8}{7} \text{ the true Quotient required.})$$

402..

358

335

239

201

Remains (38)

How such *Remainders* thus placed over their *Divisors* (which are indeed *Vulgar Fractions*) may be otherwise managed, shall be shewed farther on.

N. B. When the *Divisor* happens to be an *Unit*, viz. 1, with a *Cypher* or *Cyphers* annexed to it, as 10, 100, 1000, &c. *Division* is truly performed by cutting off with a Point or Comma, so many *Figures* of the *Dividend* as there are *Cyphers* in the *Divisor*; then are those *Figures* so cut off to be accounted a *Remainder*, and the rest of the *Figures* in the *Dividend* will be the true *Quotient* required, because an *Unit* or 1 doth neither multiply nor divide.

EXAMPLE 7.

Let it be required to divide 57842 by 100. The Work may stand thus, 100) 578,42 the *Quotient* required; or thus 100) 57842 (578,42 the same as before,

Hence it follows, that if any *Divisor* have *Cyphers* to the Right-hand of it, you may cut off so many of the last *Figures* in

in the *Dividend*, and *divide* the other *Figures* of the *Dividend*, by those *Figures* of the *Divisor* that are left when the *Cyphers* are omitted. But when *Division* is ended, those *Cyphers* so omitted in the *Divisor*, and the *Figures* cut off in the *Dividend*, are both to be restored to their own places.

EXAMPLE 8.

Suppose it were required to *divide* 675469 by 5400,

5400) 675469 (125

54..

135
108

274
560

Remains (4) But the true *Remainder* is 469.

Consequently the true *Quotient* is $125\frac{469}{5400}$.

As to the manner of proving the Truth of any Operation, either in *Multiplication* or *Division*, I presume it may be easily understood, by what is delivered in *Page 21*, compared with the three first *Examples* of *Division*; for from thence it will be easy to conceive, that if the *Divisor* and *Quotient* be multiplied together, their *Product* (with what *Remains* after *Division* being added to that *Product*) will be equal to the *Dividend*. As in the Fifth *Example*, where the *Dividend* is 379, the *Divisor* is 5, the *Quotient* is 75, and the *Remainder* is 4.

I say, $75 \times 5 = 375$, to which add the *Remainder* 4, it will be 379.

Again, in the Sixth *Example*, the *Divisor* is 67, the *Quotient* is 653, and the *Remainder* is 38.

Then $653 \times 67 = 43751$, and $43751 + 38 = 43789$ the *Dividend*, &c.

There are several useful Contractions, both in *Division* and *Multiplication*, which I have purposely omitted until I come to treat of *Decimal Arithmetick*. Also I have omitted the Business of *Evolution* or *Extracting of Roots*, until further on; and so shall conclude this *Chapter* with a few *Examples* of *Division* unwrought at large, leaving them for the Learner's Practice.

579) 43800771 (75649.
Or 75649) 43800771 (579.

45007)

45007) 23884044718 (530674.
 Or 530674) 23884044718 (45007.
 356) 244572000 (687000.
 59600) 57659066400 (967434.
 10000) 679543820000 (67954382.
 79) 282016 (3569 $\frac{5}{8}$.

C H A P. III.

Concerning Addition and Substraction of Numbers of different Denominations, and how to reduce them from one Denomination to another.

S E C T. I.

1. Of English Coin.

THE least Piece of *Money* used in *England* is a *Farthing*, and from thence ariseth the rest, as in this *Table*.

Farth.

4 = 1 d. Pen.

48 = 12 = 1 s. Skill.

960 = 240 = 20 = 1 l. Pound Sterling.

And { 5 s. is a Crown.
 10 s. is an Angel.
 6 s. 8 d. a Noble.
 13 s. 4 d. a Mark.

Note, When *l. s. d. q.* are placed over (or to the Right-hand of) *Numbers*, they denote those *Numbers* to signify *Pounds, Shillings, Pence, and Farthings*.

l. s. d. q.

As 35 10 6 2. Or 35 l. 10 s. 6 $\frac{1}{2}$ d. Either of these do signify 35 *Pounds*, 10 *Shillings*, 6 *Pence*, 2 *Farthings*.

The same must be understood of all the following *Characters*, belonging to their respective *Tables*, viz. Of *Weights, Measures, &c.*

2. Troy Weight.

The Original of all *Weights* used in *England*, was a *Corn* of *Wheat* gathered out of the middle of the *Ear*, and being well dried, 32 of them were to make one *Penny Weight*, 20 *Penny Weights* one *Ounce*, and 12 *Ounces* one *Pound Troy*. Vide *Statutes* of 51 Hen. III. 31 Edw. I. 12 Hen. VII.

But

But in later Times it was thought sufficient to divide the aforesaid *Penny Weight* into 24 equal *Parts*, called *Grains*, being the least *Weight* now in common Use; and from thence the rest are computed as in this *Table*.

Gr. Grain.					
24 =	1 P. W. Penny Weight.				
480 =	20 =	1 Oz. Ounce.			
7560 =	240 =	12 =	1 lb Pound.		

Note, { By Troy Weight are weighed Jewels, Gold, Silver, Corn, Bread, and all Liquors.

Besides the common Divisions of *Troy Weight*, I find in *Angliæ Notitia*, or, *The Present State of England*, Printed in the Year 1699, that the *Moneyers* (as that Author calls them) do subdivide the *Grain*.

Thus {	24 Blanks = 1 Periot.
	20 Periot = 1 Droite.
	24 Droites = 1 Mite.
	20 Mites = 1 Grain, &c. as before.

3. Apothecaries Weights.

The *Apothecaries* divide a *Pound Troy*, as in this *Table*.

Gr. Grain.					
20 =	1 ∅ Scruple				
60 =	3 =	1 ℥ Dram			
480 =	24 =	8 =	1 ℥ Ounce		
5760 =	288 =	96 =	12 =	1 lb Troy,	the same as before.

By these *Weights* the *Apothecaries* compound their Medicines: but buy and sell their Drugs by *Averdupois Weight*.

4. Averdupois Weight.

When *Averdupois Weight* became first in Use, or by what Law it was first settled, I cannot find out in the *Statute Books*; but on the contrary, I find that there should be but one *Weight* (and one *Measure*) used throughout this *Realm*, viz. that of *Troy*, (*Vide* 14 *Ed.* III. and 17 *Ed.* III.) So that it seems (to me) to be first introduced by *Chance*, and settled by *Custom*, viz. from giving good or large *Weight* to those *Commodities* usually weighed by it, which are such as are either very *Coarse* and *Drossy*, or very

very subject to waste; as all kind of Grocery Wares. And *Pitch, Tar, Rosin, Wax, Tallow, Flax, Hemp, &c. Copper, Tin, Steel, Iron, Lead, &c. Also Flesh, Butter, Cheese, Salt, &c.* To these and the like (I presume, it was thought convenient to allow a greater *Weight* than the Laws had provided, which happen'd to be about a Sixth part more: For I found by a very nice Experiment, that one *Pound Averdupois* is equal to 14 *Ounces*, 11 *Penny Weights*, and $15\frac{1}{2}$ *Grains Troy*. And it is now computed as in the following Table:

Drams.									
✓	16=1	Oz.	Ounces.	or	437	$\frac{75}{16}$	Grains	℔	
	256=	16=1	lb	Pounds.					
	28672=	1792=	112=	1	C.	Hundred.		And	$\left\{ \begin{array}{l} 14=a \text{ Stone} \\ 28=\frac{1}{4} \text{ of C.} \\ 56=\frac{1}{2} \text{ of C.} \\ 84=\frac{3}{4} \text{ of C.} \end{array} \right.$
	573440=	35840=	2240=	20=	1	Tun.			

5. Long Measure.

As the least part of *Weight* came at first from a *Wheat Corn*, so (it is generally said) the least part of a *Long Measure* was at first a *Barley Corn*, taken out of the middle of the Ear, and being well dried, three of them in length were to make one *Inch*; and thence the rest, as in this Table.

Barley Corns.									
	3=1	In.	Inches.			And	$\left\{ \begin{array}{l} 4 \text{ Nails} = \frac{1}{4} \text{ of a Yard.} \\ 1\frac{1}{4} \text{ Yard} = 1 \text{ Ell.} \\ 2 \text{ Yards} = 1 \text{ Fathom.} \end{array} \right.$		
	36=	12=1	F.	Feet.					
	108=	36=	3=1	Y.	Yards.				
	594=	198=	16 $\frac{1}{2}$ =	5 $\frac{1}{2}$ =1	P.	Poles.			
	23760=	7920=	660=	220=	40=1	Furlong.			
	190080=	63360=	5280=	1760=	320=8=1	Mile.			

Note, That forty *Poles* (or *Perches*) in Length, and four in Breadth, do make a Statute *Acre* of Land.

That is, 220 *Yards*, multiplied into 22 *Yards* = 4840 Square *Yards* are a Statute *Acre*.

And according to the Transactions of the *French Academy*, Anno 1687, a *Paris Foot Royal* is = 12,8 *Inches English*; Six of those Feet make a *Toise*; and 57060 *Toises* = 365184 *English Feet*, are the Measure of one Degree of a great Circle upon the Surface of the Earth. So that one Degree is 69 *Miles* and 288 *Yards*, which is very near to our Country-man Mr *Norwood's* Experiment made betwixt *London* and *York*, Anno 1635; who found that 367196 *Feet* = 69 *Miles*, and 958 *Yards* do make a

F

Degree

Degree. And not 60 *Miles*, according to the common received Opinion and Practice of the *Navigators* or *Seamen*.

Hence, according to the *French* Account, the Circumference of the Earth (supposing it to be a true *Spherical Figure*) is 24899 *English Miles*.

6. Of Liquid Measures.

All *Measures* of Capacity, both Liquid and Dry, were at first made from *Troy Weight*, *Vide Statutes 9 H. III. 51 H. III. 12 H. VII. &c.* wherein it is enacted, that eight *Pound Troy Weight*, of *Wheat*, gathered out of the middle of the Ear, and well dried, should make one *Gallon of Wine Measure*: And that there should be but one *Measure* for *Wine*, *Ale*, and *Corn*, throughout this Realm. (*Vid. Stat. 14 Ed. III. 15 Rich. II.*) But Time and Custom hath altered *Measures*, as they have done *Weights* (and perhaps for one and the same Reason) for now we have three different *Measures*, viz. one for *Wine*, one for *Ale* or *Beer*, and one for *Corn*.

I have inserted *Tables* of each, as they are now computed by *Cubick Inches*, and practised in the Art of *Gauging*, &c.

The common *Wine Gallon* sealed at *Guild-Hall* in *London*; by which all *Wines*, *Brandies*, *Spirits*, *Strong-waters*, *Mead*, *Perry*, *Cyder*, *Vinegar*, *Oil*, and *Honey*, &c. are measured and sold; is supposed to contain 231 *Cubick Inches*, and from thence the rest are computed, as in this Table.

Cubick Inches.

231	=	1	G. Gallons.
9702	=	42	= 1 Terce.
14553	=	63	= 1 $\frac{1}{2}$ = 1 Hogshead.
19404	=	84	= 2 = 1 $\frac{1}{3}$ = 1 Puncion.
29106	=	126	= 3 = 2 = 1 $\frac{1}{2}$ = 1 Butt or Pipe.
58212	=	252	= 6 = 4 = 3 = 2 = 1 Tun.

Gallons.

Note, { 18 = 1 Rundlet, and
31 $\frac{1}{2}$ makes a *Wine* or
Vinegar Barrel.
(*Vide 1 R. III.*)

But Dr *Wybard* in his *Tectometry*, Page 289, doth suppose the *Wine Gallon* to contain but 224, or 225 *Cubick Inches* at the most, and pursuant to this Account an Experiment was made by Mr *Richard Walker* and Mr *Philip Shales*, two General Officers in the Excise. They caused a Vessel to be very exactly made of *Brass*, in Form of a *Parallelopipedon*, each Side of its Base was 4 *Inches*, and its Depth 14 *Inches*; so that its just Content was 224 *Cubick Inches*. This Vessel was produced at *Guild-Hall* in *London* (May 25, 1688.) before the Lord-Mayor, the Commissioners of Excise, the Reverend Mr *Flamsteed*, Astr. Reg. Mr

Mr *Halley*, and several other ingenious Gentlemen, in whose Presence Mr *Shales* did exactly fill the aforesaid Brazen Vessel with clear Water, and very carefully emptied it into the old Standard *Wine Gallon* kept in *Guild-Hall*, which did so exactly fill it, that all then present were fully satisfied the *Wine Gallon* doth contain but 224 *Cubick Inches*. (*This notable Experiment I saw tried.*) However, for several Reasons, it was at that time thought convenient to continue the former supposed Content of 231 *Cubick Inches* to be the *Wine Gallon*, and that all Computations in *Gauging* should be made from thence, as above.

The *Beer* or *Ale Gallon* (which are both one) is much larger than the *Wine Gallon*; it being (as I presume) made at first to correspond with *Averdupois Weight*, as the *Wine Gallon* did with *Troy Weight*: For (as I said before, Page 33.) one *Pound Averdupois* is equal to 14 *Ounces* 12 *Penny Weights* *Troy*, very near.

And, as one *Pound Troy* is in proportion to the *Cubick Inches* in a *Wine Gallon*, so is one *Pound Averdupois* to the *Cubick Inches* in an *Ale Gallon*. That is, $12 : 231 :: 14\frac{1}{2} : 281\frac{1}{2}$, very near the *Cubick Inches* contained in an *Ale Gallon*, as appears from an Experiment made by one *Nicholas Gunton*, General *Gauger* in the *Excise*, about 41 Years ago, who, by such a Vessel mentioned before in the last Page, did find the Standard *Ale Quart* (kept in the *Exchequer*, *Vid.* 12 *Car.* II.) to contain just $70\frac{1}{2}$ *Cubick Inches*, consequently the *Ale Gallon* must contain 282 *Cubick Inches*, and from thence the following *Tables* are computed.

Ale-Measure.

Cubick Inches.

282 = 1 *Gallon*.
 2256 = 8 = 1 *Firkin*.
 4512 = 16 = 2 = 1 *Kilderkin*.
 9024 = 32 = 4 = 2 = 1 *Barrel*.
 13536 = 48 = 6 = 3 = $1\frac{1}{2}$ = 1 *Hogshead*.

Note, { *A Firkin of Soap and of Herrings are the same with that of Ale.*

Beer Measure.

Cub. Inches.

282 = 1 *Gallon*.
 2538 = 9 = 1 *Firkin*.
 5076 = 18 = 2 = 1 *Kilderkin*.
 10152 = 36 = 4 = 2 = 1 *Barrel*.
 15228 = 54 = 6 = 3 = $1\frac{1}{2}$ = 1 *Hogshead*.

N. B. This Distinction or Difference betwixt *Ale* and *Beer-Measure*, is now only used in *London*. But in all other Places of *England* the following *Table* of *Beer* or *Ale*, whether it be strong or small, is to be observed, according to a Statute of *Excise* made in the Year 1689

<i>Cub. Inches.</i>	
282=1	<i>Gallon.</i>
2397=8½=1	<i>Firkin.</i>
4794=17=2=1	<i>Kilderkin.</i>
9588=34=4=2=1	<i>Barrel.</i>
14382=51=6=3=1½=1	<i>Hogshead.</i>

7. Of Dry Measure.

Dry Measure is different both from *Wine* and *Ale Measure*, being as it were a Mean betwixt both, tho' not exactly so; which upon Examination I find to be in proportion to the aforesaid old Standard *Wine Gallon*, as *Averdupois Weight* is to *Troy Weight*; That is, As one *Pound Troy* is to one *Pound Averdupois*, so is the *Cubick Inches* contained in the old *Wine Gallon*, to the *Cubick Inches* contained in the *Dry* or *Corn Gallon*.

Viz. 12 : 14½ :: 224 : 272½, which is very near to 272½, the common received Content of a *Corn Gallon*: Altho' now it is otherwise settled by an Act of Parliament made in April 1697, the Words of that Act are these:

Every round Bushel with a plain and even Bottom, being made eighteen Inches and a half wide throughout, and eight Inches deep, should be esteemed a Legal Winchester Bushel, according to the Standard in his Majesty's Exchequer.

Now a Vessel being thus made will contain 2150,42 *Cubick Inches*, consequently the *Corn Gallon* doth contain but 268½ *Cubick Inches*.

<i>Cub. Inches.</i>		Note, {	4 Bushels = a Comb. 10 Quarters = a Wey, and 12 Weys = a Last of Corn.
268,8=	1 <i>Gallon.</i>		
537,6=	2=1 <i>Peck.</i>		
2150,4=	8=4=1 <i>Bushel.</i>		
17203,2=	64=32=8=1 <i>Quarter.</i>		

I observed amongst the Lead-Mines in *Derbyshire*, (*Anno* 1692) that the *Miners* bought and sold their Lead Ore, by a *Measure* which they call'd an Ore Dish; whose Dimensions I carefully took, and found it

Thus { Length 21.3.
Breadth 6.
Depth 8.4. } *Inches.*

Conse-

Consequently its Content is 1073,52 *Cubick Inches*, which is very near equal to 4 *Corn Gallons*, according to the above-mentioned Settlement.

Nine of those Dishes they call a Load of Ore, which if it be pretty good, will produce about 3 hundred Weight of Lead.

8. Of Time.

It is not an easy Thing to give a true *Definition* of *Time*; for (according to the *Philosophick Poet*)

*Time of itself is nothing, but from Thought
Receives its Rise, by labouring Fancy wrought
From Things consider'd, whilst we think on some
As present, some as past, or yet to come.
No Thought can think on Time, that's still confest,
But thinks on Things in Motion or at Rest.*

And so on, Vide *Lucretius*, Book I.

That is, *Time* only shews the *Duration* or *Mutation* of Things, a Year being the *Standard* or *Integer*, by which such Continuance or Change is computed. And a Year is that *Space* of *Time* in which the *Sun* (apparently) compleats its *Revolution* from any one *Point* in the *Ecliptick* (an imaginary *Circle* in the *Heavens*) to the same *Point* again, which, according to modern *Observations*, is performed in 365 *Days*, 5 *Hours*, 48 *Minutes*, 57 *Seconds*, 21 *Thirds*, &c. But a *Second* being the least part of *Time* that can be truly measured by the *Motion* of any *Mechanical Engine*, as a *Clock*, &c. (a *Third* being less than the *Twinkling* of an *Eye*) I begin the following Table with *Seconds*.

<i>Seconds.</i> ¹¹	
60=1'	<i>Minute.</i>
3600=	$\frac{60=1}{60=1}$ ° <i>Hour.</i>
86400=	1440= 24=1 <i>Day.</i> °
31556937=525949=8765=365+5+48+57=1 Year, called	
a Solar Year.	

But the common *Year*, usually called the *Julian Year*, doth consist of 365 *Days* and 6 *Hours*, and is divided into twelve unequal *Months*, called *Calendar Months*, whose Names and Number of *Days* are the Subject of every *Almanack*.

To these *Tables* it may not be amiss to give a brief Account of such *Coins, Weights, and Measures*, as are frequently mentioned in the Scriptures. As I have deduced them from those which seem to be the most Correct, inserted in the *Index* to the large *Bible*, Printed Anno 1702, and compared with those used in England, by the Lord Bishop of *Peterborough* [Cumberland].

The *Hebrew Weights*, compared with $\left\{ \begin{array}{l} \text{Troy Weight.} \\ \text{Oz. Pw. Gr.} \end{array} \right.$

<i>A Gerah</i> =	0 . 0 . 10 $\frac{1}{2}$
10 <i>Gerahs</i> = <i>a Bekah</i> =	0 . 4 . 13 $\frac{1}{2}$
2 <i>Bekahs</i> = <i>a Shekel</i> =	0 . 9 . 3
100 <i>Shekels</i> = <i>a Menah</i> =	45 . 12 . 12

Note, A *Shekel* is said to be their Original *Weight*.

Their Coin $\left\{ \begin{array}{l} \text{English Coin.} \\ \text{l. s. d.} \end{array} \right.$

<i>A Silver Menah</i> =	7 . 1 . 5 $\frac{1}{4}$	Weight 60 <i>Shekels</i> .
<i>Talent of Silver</i> =	357 . 11 . 10 $\frac{1}{2}$	Weight is 300 <i>Shekels</i> .
<i>Talent of Gold</i> =	5075 . 15 . 7 $\frac{1}{2}$	The same Weight men-
<i>The Gold Dram</i> =	1 . 0 . 4	tioned Ex. ii. 19.

The *Roman Money* mentioned in the *New Testament*.

<i>A Denarius</i> , or <i>Silver Penny</i> =	7 d. 3 <i>Farthings</i> .
<i>Asses of Copper</i> =	0 . 3 <i>Farthings</i> .
<i>Assarium</i> =	0 . 1 $\frac{1}{2}$ <i>Farthing</i> .
<i>Quadrans</i> =	0 . $\frac{3}{4}$ of a <i>Farthing</i> .
<i>A Mite</i> =	0 . $\frac{1}{2}$ of a <i>Farthing</i> .

Their *Long Measure*, compared with $\left\{ \begin{array}{l} \text{English Measure.} \\ \text{Yar. Feet. In. Pts.} \end{array} \right.$

<i>A Finger's Breadth</i> =	0 . 0 . 0,912
4 <i>Fingers</i> = <i>a Hand's Breadth</i> =	0 . 0 . 3,648
2 <i>Hands</i> = <i>the least Span</i> =	0 . 0 . 7,296
3 <i>Hand's Breadth</i> = <i>the longest Span</i> =	0 . 0 . 10,944
2 <i>Spans</i> = <i>the longest Cubit</i> =	0 . 1 . 9,888
4 <i>Cubits</i> = <i>a Fathom</i> =	2 . 1 . 3,552
6 <i>Cubits</i> = <i>Ezekiel's Reed</i> =	3 . 1 . 11,328
400 <i>Cubits</i> = <i>a Stadium</i> =	243 . 0 . 7,2
10 <i>Stadiums</i> = <i>a Mile</i> =	2432 . 0 . 0
3 <i>Miles</i> = <i>a Parasang</i> =	7296 . 0 . 0
Which is 4 <i>English Miles</i> and	256 .

Their

Their Measures of Capacity, compared with $\left\{ \begin{array}{l} \text{English Wine.} \\ \text{Gal. Pints. Inch.} \end{array} \right.$

<i>A Cotyla</i> =	0 . 0 $\frac{1}{2}$	3,037
<i>A Log</i> =	0 . 0 $\frac{1}{2}$	9,83
4 <i>Logs</i> = a <i>Cab</i> =	0 . 3 .	10,458
10 <i>Cotyla's</i> = an <i>Omer</i> =	0 . 6 .	1,5
3 <i>Cabs</i> = a <i>Hin</i> =	1 . 2 .	2,5
2 <i>Hins</i> = a <i>Seah</i> =	2 . 4 .	5,
3 <i>Seahs</i> = an <i>Epha</i> =	7 . 4 .	15,
10 <i>Epha's</i> = a <i>Chomer</i> =	75 . 5 .	5,625

Sect. 2. Addition of *Weights, &c.*

The foregoing *Tables* being so well understood, as that you can readily tell (without pausing) how many *Units* of any one *Denomination*, do make one of the next *Superior Denomination* (especially in those *Tables* which are most useful for your *Business*) it will then be as easy to add or subtract them, as to add or subtract whole *Numbers*, due Care being taken in placing all *Numbers* that are of one *Denomination* exactly underneath each other. That is to say, in *Money*, place *Pounds* under *Pounds*, *Shillings* under *Shillings*, *Pence* under *Pence*, &c. Understand the like in *Weights* and *Measures*, &c. according to their several *Denominations*: Then in *Addition* observe this *Rule*.

R U L E.

Always begin with those *Figures* of the lowest or least *Denomination*, and add them all together into one *Sum*, then consider how many of the next *Superior Denomination* are contained in that *Sum*, so many *Units* you must carry to the said next *Superior Denomination* to be added together with those *Figures* that stand there; and if any thing remain over or above those *Units* so carried, that *Overplus* must be set down underneath its own *Denomination*: And so proceed on from one *Denomination* to another until all be finished.

Example in Coin.

Let it be required to add 35 *l.* 14 *s.* 06 *d.* and 27 *l.* 02 *s.* 10 *d.* and 54 *l.* 13 *s.* 04 *d.* and 10 *l.* 17 *s.* 09 *d.* into one *Sum*.

The particular *Sums* being placed, as before directed, will stand as in the *Margin* following.

Then according to the *Rule*, I begin with the *Pence* (being here the lowest or least *Denomination*) and adding them all together, I find their *Sum* to be 29 *d.* that is 2 *s.* and 5 *d.* (for 24 *d.*

24 = 2s. and 29 — 24 = 5) the 5 d. I set down underneath its own *Denomination*, and carry the 2s. to the Place of *Shillings*, adding them and all the *Shillings* together, I find the *Sum* to be 48s. viz. 2l. 8s. I set down the 8s. underneath its own place of *Shillings*, and carry the 2l. to the Place of *Pounds*, adding them and all the *Pounds* together, I find their *Sum* is 128l. consequently the *Total Sum* required is 128l. 08s. 05d.

Now, for as much as it often happens in keeping Books of *Accounts*, (and in other *Business*) that it is required to add up large *Sums* of Money, consisting of 30, 40, or more several particular *Sums*, nay, perhaps, filling up the whole length of a Sheet of Paper, I humbly conceive in those Cases the best and easiest way will be to part them into *Parcels*, not exceeding above 10 or 12 particular *Sums* in each *Parcel*; that done, add together all the *Sums* of those *Parcels* into one *Sum*, and that will be the *Total Sum* required.

Also to avoid the making of *Points*, or other *Marks* amongst your *Figures*, it will be convenient to get the following *Tables* by heart,

The Pence Table.				The Shillings Table.			
d.	s.	d.	s.	s.	l.	s.	l.
12=1		72=6		20=1		120=6	
24=2		84=7		40=2		140=7	
36=3		96=8		60=3		160=8	
48=4		108=9		80=4		180=9	
60=5		120=10		100=5		200=10	

The Use of these *Tables* is so obvious, that I presume it is needless to explain them.

Examples in Addition of Weights.

Troy Weight.			
lb.	Oz.	Pw.	Gr.
3	09	00	10
5	08	15	21
10	10	12	22
0	11	19	23
Sum	21	04	09

Averdupois Weight.			
Tun.	C.	℔	Oz.
12	15	2	24
7	10	3	21
0	18	1	14
1	19	3	27
Sum	23	05	00

Examples

Examples in Addition of Long-Measure.

Yards	Qrs.	Nails	Miles	Fur.	Poles	Yards	Feet	Inch.
35	2	3	2	6	32	4	2	9
17	3	1	0	7	27	3	1	10
129	1	2	1	3	39	1	2	11

182 . 3 . 2 Sum 5 . 2 . 19 . 5 . 0 . 0

I think it needless to set down more *Examples* of this kind, for if these 5 (especially the last) be well understood, they will be sufficient to shew how any other may be performed.

Sect. 3. Subtraction of *Weights, &c.*

*S*ubtraction is but the Converse of the precedent Work, and may be performed by observing this *Rule*.

R U L E.

Begin with the Lowest or Least Denomination (as before in Addition) and Take or Subtract the Figure (or Figures) in that place of the Subtrahend, from the Figure (or Figures) that stand over them of the same Denomination; setting down the Remainder. (as in Page 12.) But if that cannot be done, then you must increase the upper Figure (or Figures) with one of the next Superior Denomination, and from that Sum make Subtraction; and so proceed to the next Superior Denomination, where you must pay the one borrowed, by adding Unity to the Subtrahend in that place, &c. as in whole Numbers.

Examples in Coin.

	l.	s.	d.		l.	s.	d.
From	386	09	08	From	569	10	06
Take	173	04	06	Subst.	389	15	08

Remains 213 . 05 . 02

179 . 14 . 10

The First of these *Examples* is self-evident. In the Second *Example*, beginning at the place of *Pence* (being here the *Least Denomination*) I am to take 8 *d.* from 6 *d.* but because that cannot be done, I must (according to the *Rule*) borrow one of the next *Denomination*, viz. 1*s.* and add it to the 6 *d.* which makes it 18 *d.* (for 1*s.* = 12 *d.* and 12 *d.* + 6 *d.* = 18 *d.* then I take 8 *d.* from that 18 *d.* and there remains 10 *d.* to be set down underneath the place of *Pence*; that done, I proceed to the place of *Shillings*, where I must now pay the 1*s.* saying one borrowed and 15 makes 16 from 10 cannot be, but

16 from 30 and there *remains* 14. That is, I borrow one of the next *Denomination*, viz. 1 *l.* and add to it the 10 *s.* which makes it 30 *s.* for 1 *l.* = 20 *s.* and 20 *s.* + 10 = 30) having set down the *Remaining* 14 *s.* underneath its own place of *Shillings*, I proceed to the place of *Pounds*, where paying the 1 *l.* borrowed, it will be 1 borrowed and 9 is 10 from 9 cannot be, but 10 from 19 and there *Remains* 9, and so on as in whole *Numbers* until all be finished; and the *Remainder* will be 179 *l.* 14 *s.* 10 *d.*

This *Example* being a little considered will render all others in this *Rule* easy.

Examples in Weights.

Troy Weight.
 lb. oz. pwt. gr.
 From 9 . 10 . 16 . 18
 Take 5 . 09 . 18 . 22

Refts 4 . 00 . 17 . 20

Averdupois Weight.
 c. qr. lb. oz.
 17 . 2 . 15 . 10
 14 . 3 . 18 . 12

2 . 2 . 24 . 14

Examples in Long Measure.

yds. qrs. nails
 From 78 . 3 . 2
 Take 29 . 3 . 3

Refts 48 . 3 . 3

miles fur. pol. yds. feet inches
 22 . 3 . 26 . 3½ . 0 . 9
 18 . 6 . 29 . 4 . 2 . 11

3 . 4 . 36 . 4 . 0 . 10

Example in Time.

days
 From 27 . 18 . 35 . 21
 Subtract 16 . 21 . 46 . 36

Remains 10 . 20 . 48 . 45

The Proof of *Addition* and *Subtraction* in these *Numbers* of different *Denominations*, is the very same with that of whole *Numbers* in *Page* 13. I shall therefore refer you to that place, and omit repeating it here.

Sect. 4. Of Reduction.

By *Reduction*, *Numbers* of different *Denominations* are brought into one *Denomination*.

That is, it alters or changes any Superior *Denomination* proposed, into any Inferior or Lesser *Denomination* Required;
 still

still keeping them equivalent in value. And by that means they become fitly prepared for *Multiplication* and *Division*; which otherwise could not so conveniently be performed. Therefore the Business of *Reduction* is very useful in the *Rule of Proportion*, (commonly called the *Golden Rule*, or *Rule of Three*) especially to those who do not understand either *Vulgar* or *Decimal Fractions*. And it is thus performed:

R U L E.

Consider how many Units of the Denomination Required, make one of that Denomination proposed to be Reduced (which is easily known by its respective Table) and with that Number of Units, Multiply the Denomination proposed, and their Product will be the Number Required.

Example in Coin.

Let it be Required to Reduce or Change 357 *l.* into *Shillings*, and those *Shillings* into *Pence*, which shall still be equal in value with the 357 *l.*

Multiply with $\begin{array}{r} 357 \\ 20 \end{array}$ the *Shillings* in one *Pound*.

Multiply with $\begin{array}{r} 7140 \\ 12 \end{array}$ = the *Shillings* in 357 *l.*
the *Pence* in one *Shilling*.

$$\begin{array}{r} 1428 \\ 714 \\ \hline \end{array}$$

85680 = the *Pence* in 357 *l.* as was Required.

Or 357 *l.* may be reduced into *Pence*, at one Operation; Thus,

Multiply with $\begin{array}{r} 357 \text{ l.} \\ 240 \end{array}$ the *Pence* contained in one *Pound*.

$$\begin{array}{r} 1428 \\ 714 \\ \hline \end{array}$$

85680 = the *Pence* in 357 *l.* as before.

But when the *Numbers* proposed to be Reduced are of several *Denominations*, and it is required to bring them all to the Lowest; you must Reduce the highest or greatest *Denomination* to the next less, Adding the *Numbers* that are of that less *Denomination* together, then Reduce their Sum to the next lower *Denomination*, Adding together all the *Numbers* that are of that *Denomination*, and so proceed gradually on 'till all is done.

G 2

EXAMPLE.

EXAMPLE.

Let it be required to Reduce 375 l. 17 s. 10 d. 3 q. into one Denomination, viz. into Farthings.

$$\begin{array}{r}
 375 \text{ l. } 17 \text{ s. } 10 \text{ d. } 3 \text{ q.} \\
 \underline{20} \\
 7500 = \text{the Shillings in } 375 \text{ l.} \\
 + 17 \text{ s.} \\
 \hline
 7517 = \text{the Shillings in } 375 \text{ l. } 17 \text{ s.} \\
 \underline{12} \\
 15034 \\
 7517 \\
 \hline
 90204 = \text{the Pence in } 375 \text{ l. } 17 \text{ s.} \\
 + 10 \text{ d.} \\
 \hline
 90214 = \text{the Pence in } 375 \text{ l. } 17 \text{ s. } 10 \text{ d.} \\
 \underline{4} \\
 360856 = \text{the Farthings in } 375 \text{ l. } 17 \text{ s. } 10 \text{ d.} \\
 + 3 \text{ q.} \\
 \hline
 360859 \text{ Farth.} = 375 \text{ l. } 17 \text{ s. } 10 \text{ d. } 3 \text{ q. as was required,}
 \end{array}$$

The Work of this Example, and all other Operations of this kind, may be somewhat shortened by observing the following Method.

$$\begin{array}{r}
 375 \text{ l. } 17 \text{ s. } 10 \text{ d. } 3 \text{ q.} \\
 \underline{20} \text{ Multiply and Add in the } 17 \text{ s.} \\
 7517 \\
 \underline{12} \text{ Multiply and Add in the } 10 \text{ d.} \\
 15034 \\
 7517 \\
 \hline
 90214 \\
 \underline{4} \text{ Multiply and Add in the } 3 \text{ qrs.} \\
 360859 \text{ the Farthings as before.}
 \end{array}$$

Example in Troy Weight.

Suppose it be Required to Reduce 29 lb. 8 oz. 18 pwt. 21 gr. into the Least Denomination, viz. into Grains.

Thus,

Thus 29 lb. 8 oz. 18 pwt. 21 gr.
 Multiply with 12 the oz. in 1 lb. and add in the 8 oz.

 66

 29

356 = the oz. in 29 lb. 8 oz.
 Multiply with 20 the pwts in 1 oz. and add in the 18 pwt.

7138 = the pwts in 29 lb. 8 oz. 18 pwt.
 Multiply with 24 the grs in 1 pwt. and add in the 21 grs.

 28553

 14278

 171333 the grs = 29 lb. 8 oz. 18 pwts. 21 grs.

These two Examples at large being well understood, may suffice to shew how all Operations of this kind are performed; either in *Weights*, *Measures*, or *Time*. I shall only insert a few Examples of each sort for the Learner's Practice.

1. In 23 C. 3 qrs. 21 lb. 9 oz. Averdupois Weight; How many Ounces? Answ. 42905 Ounces.

2. In 252 Eng. Miles, How many Yards, Feet, and Inches? Answ. 443520 yds. = 1330560 Feet = 15966720 Inches.

3. In 1692 common Years, How many Days, Hours, and Minutes? Answ. 618003 Days, 14832072 Hours, 889924320 Minutes.

Note, a common Year = 365 Days, 6 Hours, see Page 37.

4. In 5786 Pounds, 17 Shillings, 9 Pence, Sterling; How many Shillings, Pence, and Farthings? Answ. 115737 s. 1388853 d. or 5555412 Farthings. That is, 5786 l. 17 s. 9 d. = 115737 s. 9 d. = 1388853 d. &c.

The next Thing will be to shew how to bring Numbers from a lesser to a greater Denomination, which by most Authors is called (tho' very improperly)

Reduction ascending.

This Work is the Converse of the last, and is performed by Division. Thus,

R U L E.

Consider how many of the Denomination proposed make one of the Denomination required, and make that Number your Divisor, by which divide the Denomination proposed; and the Quotient will be the Number required.

E X A M P L E.

EXAMPLE.

Let it be required to find how many *Shillings* and *Pounds* are contained in 85680 *Pence*.

The *Pence* in 1*s.* are 12) 85680 ($7140s = 85680d.$

Again the *Shillings* in 1*l.* are 20) 7140 (357 *l.* the Answer required.

Another Example in Coin.

How many *Pence*, *Shillings*, and *Pounds*, are contained in 264859 *Farthings*.

12)	20)	
4) 264859	(66214 d.	(5517 s. (275 l.
24	62	151
08	21	117
05	94	(17) s.
19	(10) d.	

Remains (3) q. } Note, the Remainder is always of the same Denomination with the Dividend.

The last Quotient 275 *l.* together with the several Remainders, give the Answer required.

Viz. 275 *l.* 17 *s.* 10 *d.* 3 *q.* = 264859 *Farthings*.

Example in Troy Weight.

Suppose it were required to find how many *Pwts.* *Ozs.* and *lbs.* are contained in 171333 *Grains*.

20)	12)	
24) 171333 gr.	(7138 pw.	(356 (29 lb.
168 ...	113	24
33	138	116
24	---	108
93	(18) pws.	---
72		(8) oz.
213		
192		

Remains (21) gr.

Answ. 29 *lb.* 8 *oz.* 18 *pwt.* 21 *grs.* This and the last Example are the Reverse or Proof of those in Pages 43, 45.

1. In 42905 *Ounces*, *Averdupois weight*; How many *Pounds*, &c. Thus,

Thus 16) 42905 ²⁸⁾ (2681 lb. ⁴⁾ (95 qrs. (23 C.

$$\begin{array}{r} 109 \\ \hline 130 \\ \hline 25 \\ \hline (9) \end{array}$$

$$\begin{array}{r} 252 \\ \hline 161 \\ \hline 140 \\ \hline (21) \end{array}$$

$$\begin{array}{r} 15 \\ \hline (3) \end{array}$$

 Answ. 23 C. 3 qrs. 21 lb. 9 oz.

2. In 15966720 Inches; How many English Miles, &c.

Answ. 252 Miles, &c. as occasion requires.

There are many useful Questions may be answered by the help of *Reduction* only: As the changing of one sort of Coin for another; and comparing one sort of Measure with another, &c.

For Instance: Suppose one had 347 Rixdollars, at 4 s. 6 d. per Dollar; and desired to know how many Pounds Sterling they make,

347
 54 = the Pence in one Dollar, viz. 4 s. 6 d. = 54 d.

$$\begin{array}{r} 1388 \\ 1735 \quad 20) \\ \hline 12) 18738 d. \quad (1561 s. (78 l. \\ 67 \\ \hline 73 \\ \hline 18 \\ \hline (6) d. \end{array}$$

$$\begin{array}{r} 1561 \\ 161 \\ \hline (1) s. \end{array}$$

Answ. 78 l. 1 s. 6 d. Sterl. are = 347 Rixdollars.

Quest. 2. In 645 Flemish Ells; How many Ells English?

Note, 3 Quarters of a Yard English make one Ell Flemish, and $1\frac{1}{4}$, or 5 Quarters of a Yard, is an English Ell.

Therefore, 645

3 = the qrs of a Yard in 1 Ell Flemish.

qrs in 1 Ell = 5) 1935 (387 English Ells for the Answer.

Quest. 3. Suppose a Bill of Exchange were accepted at London, for the Payment of 400 l. Sterl. for the Value delivered at Amsterdam in Flemish Money at 1 l. 13 s. 6 d. for 1 Pound Sterl. How much Flemish Money was delivered at Amsterdam?

First, 1 l. 13 s. 6 d. = 402 d. the Value of one Pound Sterl. at Amsterdam.

Then, 402 d. $\times$ 400 = 160800 d. = 670 l. Flemish, and so much was delivered at Amsterdam.

C H A P.

C H A P. IV.

Of Vulgar Fractions.

Sect. I. Of Notation.

A *Fraction*, or *Broken Number*, is that which represents a *Part* or *Parts* of any thing proposed, (*vide* Page 3.) and is expressed by two Numbers placed one above the other with a Line drawn betwixt them :

Thus, $\left\{ \begin{array}{l} 3 \text{ Numerator.} \\ 4 \text{ Denominator.} \end{array} \right.$

The Denominator, or Number placed underneath the Line, denotes how many equal Parts the thing is supposed to be divided into (being only the Divisor in Division). And the Numerator, or Number placed above the Line, shews how many of those Parts are contained in the Fraction (it being the Remainder after Division). (*See* Page 29.) And these admit of three Distinctions:

Viz. $\left\{ \begin{array}{l} \text{Proper or Simple} \\ \text{Improper} \\ \text{Compound} \end{array} \right\} \text{Fractions.}$

A proper, pure, or *Simple Fraction*, is that which is less than an Unit. That is, it represents the immediate Part or Parts of any thing less than the whole, and therefore it's Numerator is always less than the Denominator.

As $\left\{ \begin{array}{l} \frac{1}{4} \text{ is one Fourth Part.} \\ \frac{1}{3} \text{ is one Third Part.} \end{array} \right.$ And $\left\{ \begin{array}{l} \frac{1}{2} \text{ is one Half.} \\ \frac{2}{3} \text{ is two Thirds, \&c.} \end{array} \right.$

An *Improper Fraction* is that which is greater than an Unit. That is, it represents some Number of Parts greater than the whole thing; and it's Numerator is always greater than the Denominator.

As $\frac{5}{3}$ or $\frac{7}{2}$ or $\frac{41}{13}$ &c.

A *Compound Fraction* is a Part of a Part, consisting of several Numerators and Denominators connected together with the Word [of].

As $\frac{1}{3}$ of $\frac{3}{4}$ of $\frac{2}{5}$, &c. and are thus read, The *one Third* of the *three Fourths* of the *two Fifths* of an Unit.

That is, when a Unit (or whole thing) is first divided into any Number of equal Parts, and each of those Parts are subdivided

subdivided into other Parts, and so on : Then those last Parts are called *Compound Fractions*, or *Fractions of Fractions*.

As for instance, suppose a Pound Sterling (or 20 s.) be the Unit or Whole; then is 8 s. the $\frac{2}{5}$ of it, and 6 s. the $\frac{3}{4}$ of those two Fifths, and 2 s. is the $\frac{1}{2}$ of those three Fourths; viz. 2 s. = $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{2}{5}$ of one Pound Sterling.

All *Compound Fractions* are reduced into single ones, Thus,

R U L E.

Multiply all the Numerators into one another for a Numerator, and all the Denominators into one another for the Denominator.

Thus the $\frac{1}{2}$ of $\frac{3}{4}$ of $\frac{2}{5}$ will become $\frac{6}{100}$. Or $\frac{1}{10}$.

For $1 \times 3 \times 2 = 6$ the Numerator, and $3 \times 4 \times 5 = 60$ the Denominator, but $\frac{6}{60}$ or $\frac{1}{10}$ of a l. Sterl. is 2 s. As above.

Sect. 2. To Alter or Change different Fractions into one Denomination retaining the same Value.

IN order to gain a clear Understanding of this Section, it will be convenient to premise this Proposition, viz. If a Number multiplying two Numbers produce other Numbers, the Numbers produced of them shall be in the same proportion that the Numbers multiplied are, 17 *Euclid* 7.

That is to say, If both the Numerator and Denominator of any *Fraction* be equally multiplied into any Number, their Products will retain the same Value with that Fraction.

As in these, $\frac{2 \times 2}{3 \times 2} = \frac{4}{6}$. Or $\frac{2 \times 3}{3 \times 3} = \frac{6}{9}$. Or $\frac{2 \times 5}{3 \times 5} = \frac{10}{15}$, &c.

That is, $\frac{2}{3}$ and $\frac{4}{6}$. Or $\frac{2}{3}$ and $\frac{6}{9}$. Or $\frac{2}{3}$ and $\frac{10}{15}$ are of the same Value, in respect to the Whole or Unit.

From hence it will be easy to conceive, how two or more Fractions that are of different Denominations, may be altered or changed into others that shall have one common Denominator, and still retain the same Value.

Example. Let it be required to change $\frac{2}{3}$ and $\frac{3}{4}$ into two other Fractions that shall have one common Denominator, and yet retain the same Value.

According to the foregoing Proposition, if $\frac{2}{3}$ be equally multiplied with 7, it will become $\frac{14}{21}$, viz. $\frac{2 \times 7}{3 \times 7} = \frac{14}{21}$. Again, if $\frac{3}{4}$ be equally multiplied with 3, it will become $\frac{9}{12}$, viz. $\frac{3 \times 3}{4 \times 3} = \frac{9}{12}$.

H

And

And by this means I have obtained two new Fractions, $\frac{1}{2}\frac{4}{1}$ and $\frac{2}{2}\frac{2}{1}$, that are of one Denomination, and of the same Value with the two first proposed, viz. $\frac{1}{2}\frac{4}{1} = \frac{2}{1}$ and $\frac{2}{2}\frac{2}{1} = \frac{2}{1}$.

And from hence doth arise the general Rule for bringing all Fractions into one Denomination.

R U L E.

Multiply all the Denominators into each other for a new (and common) Denominator. And each Numerator into all the Denominators but it's own, for new Numerators.

Example. Let the proposed Fractions be $\frac{1}{3}$, $\frac{2}{3}$, $\frac{3}{4}$, and $\frac{6}{7}$.

Then, by the Rule,

A new Denominator
will be thus found.

$$\begin{array}{r} 3 \\ \times 5 \\ \hline 15 \\ \times 4 \\ \hline 60 \\ \times 7 \\ \hline 420 \end{array}$$

And the new Numerators will
be thus found.

1.	2.	3.	6
$\frac{1}{5}$	$\frac{2}{3}$	$\frac{3}{3}$	$\frac{6}{3}$
$\frac{5}{5}$	$\frac{6}{6}$	$\frac{9}{9}$	$\frac{18}{18}$
$\frac{4}{4}$	$\frac{4}{4}$	$\frac{5}{5}$	$\frac{5}{5}$
$\frac{20}{20}$	$\frac{24}{24}$	$\frac{45}{45}$	$\frac{90}{90}$
$\frac{7}{7}$	$\frac{7}{7}$	$\frac{7}{7}$	$\frac{4}{4}$
140.	168.	315.	360

Hence 420 is the common Denominator; and 140 . 168 . 315 . 360, are the new Numerators, which being placed Fraction-wise are $\frac{140}{420}$. $\frac{168}{420}$. $\frac{315}{420}$. $\frac{360}{420}$, the New Fractions required.

That is, $\frac{140}{420} = \frac{1}{3}$. $\frac{168}{420} = \frac{2}{5}$. $\frac{315}{420} = \frac{3}{4}$ and $\frac{360}{420} = \frac{6}{7}$.

Sect. 3. To bring mix'd Numbers into Fractions, and the contrary.

MIX'D Numbers are brought into improper Fractions by the following Rule.

R U L E.

Multiply the Integers, or whole Numbers, with the Denominator of the given Fraction, and to their Product add the Numerator, the Sum will be the Numerator of the Fraction required.

Example. $9\frac{4}{5}$ by the Rule will become $\frac{49}{5}$. For $9 \times 5 = 45$. And, $45 + 4 = 49$, the improper Fraction required.

Again, $13\frac{1}{3}$ will become $\frac{40}{3}$. For $13 \times 3 = 39$.

And $\frac{40}{3} + \frac{1}{3} = \frac{41}{3}$. And so for any other as occasion requires.

To find the true Value of any improper Fraction given is only the Converse of this Rule. For if $\frac{49}{5} = 9\frac{4}{5}$, as before is evident :

*

Then

Then it follows that if 49 be divided by 5, the Quotient will give $9\frac{4}{5}$. And if 206 be divided by 15, it will give $13\frac{11}{15}$, &c. consequently it follows, that

If the Numerator of any improper Fraction be divided by it's Denominator, the Quotient will discover the true Value of that Fraction.

E X A M P L E S.

$\frac{49}{5} = 9\frac{4}{5}$. And $\frac{206}{15} = 13\frac{11}{15}$, And $\frac{120}{20} = 6\frac{1}{2}$, Or $\frac{12}{4} = 3\frac{1}{4}$, &c.

When whole Numbers are to be expressed Fraction-wise, it is but giving them an Unit for a Denominator. Thus 45 is $\frac{45}{1}$, 9 is $\frac{9}{1}$, and 25 is $\frac{25}{1}$, &c.

Sect. 4. To Abbreviate or Reduce Fractions into their Lowest or Least Denomination.

THIS is done, not out of any necessity, but for the more convenient managing of such *Fractions* as are either proposed in large terms; or swell into such, either by *Addition* or otherwise: besides it is most like an Artist to express or set down all *Fractions* in the lowest terms possible; and to perform that, it will be necessary to consider these following Propositions.

Numbers are either Prime or Composed.

1. A Prime Number is that which can only be measured by an Unit. *Euclid 7. Defin. 11.*

That is, 3, 5, 7, 11, 13, 17, &c. are said to be Prime Numbers, because it is not possible to divide them into equal Parts by any other Number but Unity or 1.

2. Numbers Prime the one to the other, are such as only an Unit doth Measure, being their common Measure. *Euclid 7. Defin. 12.*

For instance, 7 and 13 are Prime Numbers to each other, because they cannot be divided by any Number but an Unit. And 9 and 14 are also Prime Numbers to each other, for altho' 3 will measure or divide 9 without leaving a Remainder; yet 3 will not measure 14 without leaving a Remainder: Again, altho' 2 will measure 14 without any Remainder, yet 2 will not measure 9 without leaving a Remainder, &c.

3. A composed Number is that which some certain Number measureth. *Euclid 7. Defin. 13.*

For instance, 15 is a composed Number of 3 and 5, for $5 \times 3 = 15$, consequently 3 or 5 will justly measure 15. Also 20

is composed of 5 and 4, viz. $5 \times 4 = 20$, therefore 5 and 4 will each justly measure 20.

4. Numbers composed the one to the other, are they which some Number, being a common Measure to them both, doth measure. *Euclid 7. Defin. 14.*

That is, if two or more Numbers can be divided by one and the same Divisor; then are those Numbers said to be composed one to another,

For Instance, 14 and 21 are Numbers composed the one to the other, because they can both be measured or divided by 7. For $7 \times 2 = 14$, and $7 \times 3 = 21$; therefore 7 is a common Measure to 14 and 21. So that if $\frac{14}{21}$ were proposed to be abbreviated, it will become $\frac{2}{3}$.

$$\text{Thus } \left\{ \begin{array}{l} 7) 14 = 2 \\ 7) 21 = 3 \end{array} \right.$$

And how those greatest common Measures may be found comes from *Euclid 7. Prob. 1, 2, 3*, and is thus:

R U L E.

Divide the greater Number by the lesser, and that Divisor by the Remainder (if there be any), and so on continually until there be no Remainder left: Then will that last Divisor be the greatest common Measure (and if it happen to be 1, then are those Numbers Prime Numbers, and are already in their lowest Terms; but if otherwise) Divide the Numbers by that last Divisor, and their Quotients will be their least Terms required.

E X A M P L E.

Let it be required to find the greatest common Measure of 72 and 108, viz. of $\frac{72}{108}$.

$$\begin{array}{r} 72) 108 \text{ (1)} \\ \underline{72} \\ 36) 72 \text{ (2)} \\ \underline{72} \\ (0) \end{array} \quad \left\{ \begin{array}{l} \text{Here because there is no Remainder;} \\ 36 \text{ is the greatest common Measure.} \end{array} \right.$$

Therefore, $\left\{ \begin{array}{l} 36) 72 = 2 \\ 36) 108 = 3 \end{array} \right\}$ Hence $\frac{72}{108}$ is abbreviated to $\frac{2}{3}$ the lowest Terms.

Again, to find the greatest common Measure of 744 and 899.

Thus,

$$\begin{array}{r}
 \text{Thus, } 744) 899 \text{ (1)} \\
 \underline{744} \\
 155) 744 \text{ (4)} \\
 \underline{620} \\
 124) 155 \text{ (1)} \\
 \underline{124} \\
 31) 124 \text{ (4)} \\
 \underline{124} \\
 (0)
 \end{array}$$

Here 31 is found to be the greatest common Measure by which 744 and 899 may be abbreviated to 24 and 29 their lowest Terms.

Thus, $\frac{29}{31} \frac{744}{899} (= \frac{24}{29}, \&c.$

Note, If the proposed Numbers be even, they may be brought lower by a continued halving of them, so long as they can be halved, *viz.* divided by 2.

E X A M P L E.

It is required to Reduce $\frac{56}{34}$ to it's least Terms.

First, $\frac{2}{2} \frac{56}{34} (= \frac{28}{17})$. Again, $\frac{2}{2} \frac{28}{17} (= \frac{14}{17})$.

This done, you easily perceive that 7 will be the common Measure to 14 and 17, *viz.* $\frac{2}{7} \frac{14}{17} (= \frac{2}{17}, \&c.$

If the Numbers proposed to be reduced have each a Cypher, or Cyphers annexed to them, they will be abbreviated by cutting off a like Number of Cyphers from both.

Thus, $\frac{1500}{300}$ will be $\frac{15}{3}$. And $\frac{2000}{300}$ will be $\frac{20}{3}, \&c.$

That is, $\frac{1500}{300} = \frac{15}{3} = \frac{5}{1}$. And $\frac{2000}{300} = \frac{20}{3}$. And $\frac{3600}{400} = \frac{36}{4} = \frac{9}{1}$.

Sect 5. Addition of Fractions.

WHAT hath been done by the Rules in this Chapter, is chiefly to prepare and fit *Fractions* of different Denominations for *Addition* or *Subtraction*, as Occasion requires, *viz.* If they are *Compound Fractions*, they must be reduced to Simple or Pure *Fractions*, *per Rule, Sect. 1.*

If they are of different Denominations, they must be altered or changed, *per Rule, Sect. 2.*

That is, all *Fractions* must be brought into one Denomination before they can either be added or subtracted; and that being done, *Addition* is thus performed.

R U L E.

Add together all the Numerators, and their Sum will be a New Numerator, under which subscribe the Common Denominator.

Examples

Examples in Simple Fractions.

Let it be proposed to add $\frac{1}{3}$, $\frac{2}{5}$, and $\frac{3}{4}$ together. First, $\frac{1}{3} = \frac{20}{60}$, $\frac{2}{5} = \frac{24}{60}$, and $\frac{3}{4} = \frac{45}{60}$, per Sect. 2.

Then $\frac{20}{60} + \frac{24}{60} + \frac{45}{60} = \frac{89}{60}$, the Sum required, which according to Section 3, is $1 \frac{29}{60}$, viz. $\frac{89}{60} = 1 \frac{29}{60}$.

Examples in Compound Fractions.

Let it be required to add $\frac{3}{7}$ and $\frac{2}{3}$ of $\frac{1}{4}$ into one Sum. First $\frac{2}{3}$ of $\frac{1}{4}$ becomes $\frac{6}{12}$ or $\frac{1}{2}$ per Sect. 1. And (per Sect. 2.) $\frac{3}{7}$ and $\frac{1}{2}$ is $\frac{6}{14}$ and $\frac{7}{14}$, viz. $\frac{3}{7} = \frac{6}{14}$, and $\frac{1}{2} = \frac{7}{14}$; but $\frac{6}{14} + \frac{7}{14} = \frac{13}{14}$ the Sum required, viz. $\frac{3}{7} + \frac{2}{3}$ of $\frac{1}{4} = \frac{13}{14}$.

Examples in mixed Numbers.

It is required to add $5 \frac{2}{3}$ to $7 \frac{3}{4}$, these per Sect. 3. will be $\frac{17}{3}$ and $\frac{31}{4}$. But $\frac{17}{3}$ and $\frac{31}{4}$ will become $\frac{68}{12}$ and $\frac{93}{12}$ per Sect. 2. Then $\frac{68}{12} + \frac{93}{12} = \frac{161}{12}$, and $\frac{161}{12} = 13 \frac{5}{12}$ the Sum required.

Or you may bring only the *Fractions* to one Denomination.

Thus, $5 \frac{2}{3}$ and $7 \frac{3}{4}$ will become $5 \frac{8}{12}$ and $7 \frac{9}{12}$.

Then $5 \frac{8}{12} + 7 \frac{9}{12} = 12 \frac{17}{12}$. That is, $13 \frac{5}{12}$. As before.

Sect. 6. *Subtraction of Fractions.*

R U L E.

SUBTRACT one Numerator from the other (according as the Question requires) and their Difference will be a new Numerator, under which subscribe the Common Denominator, as in Addition.

E X A M P L E 1.

Let it be required to take $\frac{2}{9}$ out of $\frac{3}{7}$. First $\frac{2}{9}$ and $\frac{3}{7}$ per Sect. 2. will become $\frac{14}{63}$ and $\frac{27}{63}$; then $\frac{27}{63} - \frac{14}{63} = \frac{13}{63}$, that is, $\frac{3}{7} - \frac{2}{9} = \frac{13}{63}$. As was required.

E X A M P L E 2.

It is required to subtract $\frac{2}{3}$ of $\frac{8}{9}$ from $\frac{13}{4}$. First, $\frac{2}{3}$ of $\frac{8}{9} = \frac{16}{27}$ per Sect. 1. Again $\frac{16}{27}$ and $\frac{13}{4}$ will become $\frac{224}{378}$ and $\frac{351}{378}$. per Sect. 2. Then $\frac{351}{378} - \frac{224}{378} = \frac{127}{378}$.

E X A M P L E 3.

From $6 \frac{1}{8}$ subtract $3 \frac{1}{4}$. First, $6 \frac{1}{8} = \frac{49}{8}$, and $3 \frac{1}{4} = \frac{13}{4}$ per Rule Sect. 3. Again, $\frac{49}{8} = \frac{245}{384}$, and $\frac{13}{4} = \frac{1304}{384}$, per Rule Sect. 2. Then, $\frac{245}{384} - \frac{1304}{384} = \frac{1059}{384} = 2 \frac{280}{384} = 2 \frac{35}{48}$. Or otherwise thus:
First,

First, $6\frac{1}{8} = 5\frac{8}{8}$, then bring $\frac{9}{8}$ and $\frac{10}{8}$ into one Denomination, viz.
 $5\frac{8}{8} = 5\frac{17}{8}$ and $3\frac{10}{8} = 3\frac{15}{8}$.

Then $5\frac{17}{8} - 3\frac{15}{8} = 2\frac{2}{8} = 2\frac{1}{4}$. As before.

E X A M P L E.

Let it be required to subtract $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$ from 7.

First, $\frac{3}{7}$ of $\frac{5}{9}$ of $\frac{2}{3} = \frac{2}{27}$. And $7 = 6\frac{18}{27}$.

Then $6\frac{18}{27} - \frac{2}{27} = 6\frac{16}{27} = 6\frac{5}{9} = 7 - \frac{1}{7}$ of $\frac{5}{9}$ of $\frac{2}{3}$. As was required.

If these few *Examples* be well understood, the whole Business of adding and subtracting *Vulgar Fractions* will be easy; which is really much more difficult to perform than either *Multiplication* or *Division*; as will appear in the next *Section*.

SECT. 7. Multiplication of Fractions.

IN order to perform either *Multiplication* or *Division*, you must prepare the Terms to be multiplied (or divided) thus; reduce Compound *Fractions* to Simple ones, per *SECT. 1*. Bring mixed Numbers into improper *Fractions*, and express whole Numbers *Fraction-wise*, per *SECT. 3*. Also it will be convenient to abbreviate them to their smallest Terms, when it can be done. Then *Multiplication* may be thus performed.

Rule. $\left\{ \begin{array}{l} \text{Multiply the Numerators one into another for a new Nu-} \\ \text{merator; and the Denominators one into another for a new} \\ \text{Denominator. As in these} \end{array} \right.$

E X A M P L E S.

1. The Product of $\frac{2}{5}$ into $\frac{3}{7} = \frac{6}{35}$. That is, $\frac{2 \times 3}{5 \times 7} = \frac{6}{35}$.
2. And the Products of $\frac{9}{18}$ into $\frac{20}{27} = \frac{180}{486}$. Or $\frac{5}{12}$.
3. Again, the Product of $\frac{7}{11}$ into $\frac{2}{3}$ of $\frac{5}{7} = \frac{70}{385}$. Or $\frac{2}{11}$.
 For $\frac{2}{3}$ of $\frac{5}{7} = \frac{10}{21}$. Then $\frac{7}{11} \times \frac{10}{21} = \frac{70}{231} = \frac{2}{11}$.
4. Let it be required to multiply 6 with $3\frac{2}{5}$. These prepared for the Work will stand thus. $\frac{6}{1} \times \frac{17}{5}$.
 viz. $6 = \frac{6}{1}$ and $3\frac{2}{5} = \frac{17}{5}$. Then $\frac{6}{1} \times \frac{17}{5} = \frac{102}{5}$, or $20\frac{2}{5}$.
 Or, otherwise thus $6 \times 3 = 18$. And $\frac{2}{5} \times 6 = \frac{12}{5} = 2\frac{2}{5}$.
 Then $18 + 2\frac{2}{5} = 20\frac{2}{5}$. As before.
5. Let it be required to multiply $7\frac{4}{9}$ with $5\frac{3}{7}$.
 First $7\frac{4}{9} = \frac{67}{9}$ and $5\frac{3}{7} = \frac{38}{7}$. Then $\frac{67}{9} \times \frac{38}{7} = \frac{2546}{63} = 40\frac{22}{63}$.

Now the Reason of this Rule for multiplying of *Fractions*, and consequently of these Operations, and all others performed by it; will be evident from this following.

Viz.

Viz. If $\frac{4}{2}$ be multiplied with $\frac{1}{3}$ according to the Rule, their Product will be $\frac{4}{6}$. But $\frac{4}{6} = \frac{2}{3}$.

Now $\frac{4}{2} = 2$, and $\frac{1}{3} = \frac{1}{3}$ per Sect. 3. But $4 \times 2 = 8$. *Ergo*, &c.

Sect. 8. Division of Fractions.

THE *Fractions* being first prepared as before directed, *Division* may be thus performed:

Rule. $\left\{ \begin{array}{l} \text{Multiply the Numerator of the Dividend into the Denominator of the dividing Fraction for a new Numerator: and} \\ \text{multiply the other Numerator and Denominator together for a} \\ \text{new denominator.} \end{array} \right.$

E X A M P L E S.

1. Let $\frac{6}{35}$ be divided by $\frac{3}{7}$, *viz.* $\frac{3}{7} \frac{6}{35} (\frac{42}{105} = \frac{2}{5}$ the Quotient.
That is, according to the Rule $6 \times 7 = 42$ the new Numerator, and $35 \times 3 = 105$, the new Denominator, &c. as above.
2. Let it be required to divide $\frac{20}{27}$ by $\frac{5}{12}$, *viz.* $\frac{5}{12} \frac{20}{27} (\frac{240}{135} = 1 \frac{7}{9}$.
For $12 \times 20 = 240$ the new Numerator, and $27 \times 5 = 135$ the new Denominator.
3. Suppose it were required to divide $\frac{2}{11}$ by $\frac{2}{5}$ of $\frac{5}{7}$.
First, $\frac{2}{5}$ of $\frac{5}{7} = \frac{10}{35}$. Then $\frac{10}{35} \frac{2}{11} (\frac{20}{385} = \frac{4}{77}$.
4. Let $20 \frac{2}{5}$ be divided by $3 \frac{2}{5}$; *viz.* $\frac{102}{5}$ by $\frac{17}{5}$:
For $20 \frac{2}{5} = \frac{102}{5}$, and $3 \frac{2}{5} = \frac{17}{5}$. Then $\frac{17}{5} \frac{102}{5} (= 6$ the Quotient.
5. Let it be required to divide $40 \frac{26}{63}$ by $5 \frac{3}{7}$.
First, $40 \frac{26}{63} = \frac{2546}{63}$, and $5 \frac{3}{7} = \frac{38}{7}$. Then $\frac{38}{7} \frac{2546}{63} (\frac{17822}{441}$.
But $\frac{17822}{441} = 7 \frac{4}{9}$ the true Quotient required.
6. Suppose it were required to divide 13 by $\frac{5}{7}$.
First, $13 = \frac{13}{1}$. Then $\frac{5}{7} \frac{13}{1} (\frac{65}{7} = 9 \frac{2}{7}$, the Quotient.
7. Again, let it be required to divide $\frac{5}{7}$ by 6 .
Viz. $\frac{6}{1} \frac{5}{7} (\frac{5}{7}$ for the Quotient required.

N. B. From hence you may observe, that when any whole Number is divided by a Fraction less than Unity or 1, the Quotient will be greater than the Number proposed to be divided: But if any Fraction be divided by a whole Number, greater than 1, then the Quotient will be less than the Dividend: As in the two last *Examples*.

As to the *Reason* (or *Proof*) of this *Rule* for dividing *Fractions*: It is only the *Converse* to that of *Multiplication*, and will be very evident from this following.

Let $\frac{4}{6}$ be divided by $\frac{1}{2}$. Which according to the *Rule* is thus, $\frac{4}{6} \div \frac{1}{2} = \frac{4}{6} \times \frac{2}{1} = \frac{8}{3} = 2\frac{2}{3}$. The true *Quotient*. Now $\frac{4}{6} = \frac{2}{3}$. And $\frac{2}{3} \div \frac{1}{2} = \frac{2}{3} \times \frac{2}{1} = \frac{4}{3} = 1\frac{1}{3}$. *per Sect. 3.* Consequently $\frac{4}{6}$ divided by $\frac{1}{2}$ is but the same with 8 divided by 2; viz. 2) 8 (4. The *Quotient* as before.

I could have inserted *Geometrical Demonstrations*, for the *Rules* of *Multiplication* and *Division* of *Fractions*; but supposing the *Learner* purely unacquainted with those kind of *Demonstrations*, I thought these might be more intelligible to him, especially in this place.

CHAP. V.

Of Decimal Fractions.

WHEN, or by whom, this excellent *Invention* of *Decimal Arithmetick*, was first introduced is uncertain; but doubtless it's *Improvements*, and the *Perfections* it is now in, is owing to latter *Years*.

Sect. I. Of Notation.

IN *Decimal Fractions*, the *Integer* or *whole Thing* (whether it be *Coin*, *Weight*, *Measure*, or *Time*, &c.) is supposed to be divided into *Ten equal Parts*; and every one of those *Ten Parts* are supposed to be subdivided into other *Ten equal Parts*, &c. *ad infinitum*.

The *Integer* being thus divided (by *Imagination*) into 10, 100, 1000, 10000, &c. equal *Parts*, becomes the *Denominator* to the *Decimal Fractions*.

Thus, $\frac{1}{10}$. $\frac{1}{100}$. $\frac{1}{1000}$. $\frac{1}{10000}$. $\frac{1}{100000}$. &c.

Now these *Denominators* are seldom or never set down, but only the *Numerators*; and those are either distinguished, or separated from *whole Numbers* by a *Point* or a *Comma*.

Thus, 5,4 is 5 $\frac{4}{10}$. and 0,7 is $\frac{7}{10}$. 35,05 is 35 $\frac{5}{100}$, &c.

But before we proceed further in *Notation*, it will be convenient for the *Learner* to consider the following *Table*, (taken out of the learned Mr *Oughtred's Clavis Mathematica*) which shews the very *Foundation* of *Decimal Fractions*.

I

Whole

Whole Numbers,						Decimal Parts.					
5	4	3	2	1	0	1	2	3	4	5	6
<i>Units Place.</i> <i>Tens.</i> <i>Hundreds.</i> <i>Thousands.</i> <i>Tens of Thousands.</i> <i>&c.</i>						<i>Parts of a Million.</i> <i>Parts of 100 Thousand.</i> <i>Parts of Ten Thousand.</i> <i>Parts of a Thousand.</i> <i>Parts of a Hundred.</i> <i>Parts of Ten, or $\frac{1}{10}$.</i> <i>&c.</i>					

By this *Table* it is evident, that as in whole *Numbers* or *Integers*, every *Degree* from the *Units Place* increases towards the left-hand by a *Ten-fold Proportion*: So in *Decimal Parts* every *Degree* is decreased towards the right-hand by the same *Proportion*, viz. by *Tens*.

Therefore these *Decimal Parts* or *Fractions*, are really more *Homogeneous*, or agreeing with whole *Numbers*, than *Vulgar Fractions*; for indeed all plain *Numbers* are in effect but *Decimal Parts* one to another.

That is, suppose any *Series* of equal *Numbers*, as 444, &c. The first 4 towards the Left is *Ten* times the *Value* of the 4 in the middle, and that 4 in the middle is *Ten* times the *Value* of the last 4 to the Right of it, and but the *Tenth Part* of that 4 on the left, &c.

Therefore all or any of them may be taken either as *Integers*, or *Parts* of an *Integer*: If *Integers*, then they must be set down without any *Comma* or separating *Point* betwixt them thus, 444 But if *Integers*, and one *Part* or *Fraction*, put a *Comma* betwixt them thus, 44,4 which signifies 44 whole *Numbers*, and 4 *Tenths* of an *Unit*: Again, if two *Places* of *Parts* be required, separate them with a *Comma* thus, 4,44 viz. 4 *Units*, and 44 *hundred Parts* of an *Unit*, &c.

From hence (duly compared with the *Table*) it will be easy to conceive that *Decimal Parts* take their *Denomination* from the *Place* of their last *Figure*.

$$\text{That is, } \left\{ \begin{array}{l} ,5 = \frac{5}{10} \\ ,56 = \frac{56}{100} \\ ,056 = \frac{56}{1000} \end{array} \right\} \text{Parts of an Unit, \&c.}$$

Cyphers

Cyphers annexed to Decimal Parts, alter not their Value. As ,50, and ,500, or ,5000, &c. are each but 5 Tenths of an Unit. For $\frac{5}{100} = \frac{5}{10}$. And $\frac{5}{1000} = \frac{5}{10}$. Or $\frac{5}{10000} = \frac{5}{10}$. Per Sect. 4. of the last Chapter.

But *Cyphers prefixed to Decimal Parts decrease their Value, by removing them further from the Comma.*

Thus, $\left\{ \begin{array}{l} ,5 = 5 \text{ Tenth Parts.} \\ ,05 = 5 \text{ Parts of a Hundred.} \\ ,005 = 5 \text{ Parts of a Thousand.} \\ ,0005 = 5 \text{ Parts of Ten Thousand, \&c.} \end{array} \right.$

Consequently the true *Value* of all *Decimal Parts* are known by their Distance from the *Units Place*; the which being once rightly understood, the rest will be easy.

Sect. 2. Addition and Substraction of Decimals.

IN setting down the proposed Numbers to be added, or subtracted, great care must be taken in placing every Figure directly underneath those of the same Value, whether they be mix'd Numbers, or pure Decimal Parts, and to perform that you must have a due regard to the Comma's, or separating Points, which ought always to stand in a direct Line one under another; and to the Right-hand of them carefully place the Decimal Parts, according to their respective Values, or Distances from Unity. Then

Rule $\left\{ \begin{array}{l} \text{Add or subtract them, as if they were all whole Numbers;} \\ \text{and from their Sum, or Difference, cut off so many Decimal} \\ \text{Parts as are the most in any of the given Numbers.} \end{array} \right.$

EXAMPLES in Addition.

Let it be required to find the Sum of these following Numbers, viz. $34,5 + 65,3 + 128,7 + 95 + 87,8 + 7,9$, which being truly placed, will stand

$$\begin{array}{r} \text{Thus, } \left\{ \begin{array}{r} 34,5 \\ 65,3 \\ 128,7 \\ 95,0 \\ 87,8 \\ 7,9 \\ \hline \end{array} \right. \\ \text{Their Sum required, } 419,2 \end{array}$$

E X A M P L E 2.

Let it be required to find the Sum of $25,854 + 34,578 + 9,076 + 13,907$.

$$\begin{array}{r}
 25,854 \\
 34,578 \\
 9,076 \\
 13,907 \\
 \hline
 83,415 \text{ The Sum required.}
 \end{array}$$

When the Decimal Parts proposed to be added (or subtracted) have not the same Number of Places, you may for convenience of Operation supply or fill up the void Places, by annexing Cyphers. As in these *Examples*.

EXAMPLE 3.

$$\begin{array}{r}
 45,0700 \\
 50,7580 \\
 123,0057 \\
 74,7020 \\
 24,8000 \\
 \hline
 318,3357
 \end{array}$$

EXAMPLE 4.

$$\begin{array}{r}
 574,678953 \\
 95,796430 \\
 78,054600 \\
 54,789000 \\
 8,900000 \\
 \hline
 812,218983
 \end{array}$$

EXAMPLE 5.

$$\begin{array}{r}
 0,975642 \\
 ,745257 \\
 ,000598 \\
 ,800700 \\
 ,640530 \\
 \hline
 3,162727
 \end{array}$$

E X A M P L E S in Subtraction.

Let it be required to find the Difference between 45,375 and 74,284.

EXAMPLE 1.

That is, $\left\{ \begin{array}{l} \text{From } 74,284 \\ \text{Take } 45,375 \end{array} \right.$

$$\begin{array}{r}
 \text{Remains } 28,909
 \end{array}$$

EXAMPLE 2.

From 437,5
Take 89,657

$$\begin{array}{r}
 347,843
 \end{array}$$

EXAMPLE 3.

From 75,0034
Take 57,875

$$\begin{array}{r}
 17,1284
 \end{array}$$

E X A M P L E 4.

Let it be required to find the Excess between 562 and 93,5784.

EXAMPLE 4.

That is, $\left\{ \begin{array}{l} \text{From } 562, \\ \text{Take } 93,5784 \end{array} \right.$

$$\begin{array}{r}
 \text{The Excess } 468,4216
 \end{array}$$

EXAMPLE 5.

From 345,7578
Take 157,

$$\begin{array}{r}
 188,7578
 \end{array}$$

Note, The two last Examples are supposed to be supplied with Cyphers, which if actually done would stand thus,

$$\begin{array}{r}
 562,0000 \\
 93,5784 \\
 \hline
 \text{Remains } 468,4216
 \end{array}$$

As before,

$$\begin{array}{r}
 345,7578 \\
 157,0000 \\
 \hline
 188,7578
 \end{array}$$

EXAMPLE

EXAMPLE 6.

From 0,547893
Take 0,439758
0,108135

EXAMPLE 7.

From 1,000000
Take 0,997543
0,002457

The Proof of Addition and Subtraction in Decimals, is the same with that of whole Numbers, page 13, &c.

Sect. 3. Multiplication of Decimals.

WHETHER the Factors or Numbers to be multiplied are pure Decimals, or mixed. Multiply them as if they were all whole Numbers, and for the true Value of their Product observe this

Rule { *Cut off (viz. separate with a Comma) so many Places of Decimal Parts in the Product, as there are in both the Factors accounted together. As in these :*

EXAMPLE 1.

3,024
2,23
90 72
604 8
6 048
6,743 52

EXAMPLE 2.

32,12
24,3
9 63 6
128 48
642 4
780,51 6

The Reason why such a Number of Decimal Parts must be cut off in the Product, may be easily deduced from these Examples. Thus,

In Example 1. It is evident, that 3, the whole Number in the Multiplicand, being multiplied with 2, the whole Number in the Multiplier; can produce but 6 (*viz.* $3 \times 2 = 6$). So that of necessity all the other Figures in the Product must be Decimal Parts; according as the Rule directs.

Or, the Rule is evident from the Multiplication of whole Numbers only: Thus, suppose 3000 were to be multiplied with 200, their Product will be 600000; That is, there will be so many Cyphers in the Product, as are in both the Factors. (*Vide page 18.*) Now if, instead of those Cyphers in the Factors, we suppose the like Number of Decimal Parts; then it follows, that there ought to be the same Number of Decimal Parts in the Product, as there were Cyphers in the Factors.

Again, the Rule may be otherwise made evident from Vulgar Fractions, thus: Let 32,12 be multiplied with 24,3, and

and their Product will be 780,516 as in *Example 2*, above. Now $32,12 = 32 \frac{12}{100}$. and $24,3 = 24 \frac{3}{10}$. which being brought into Improper Fractions (*per Sect. 3. page 50.*) will become $32 \frac{12}{100} = \frac{3212}{100}$. and $24 \frac{3}{10} = \frac{243}{10}$.

Then $\frac{3212}{100} \times \frac{243}{10} = \frac{780516}{1000}$. *per Sect. 7. page 55.*

But $\frac{780516}{1000} = 780 \frac{516}{1000}$. viz. 780,516, as before.

Any of these three Ways do, I presume, sufficiently prove the Truth of the abovesaid Rule, &c.

EXAMPLE 3.

$$\begin{array}{r} 78,546 \\ 436 \\ \hline 471276 \\ 235638 \\ 314184 \\ \hline 34246,056 \end{array}$$

EXAMPLE 4.

$$\begin{array}{r} 5745 \\ ,0675 \\ \hline 28725 \\ 40215 \\ 34470 \\ \hline 387,7875 \end{array}$$

N. B. It sometimes falls out in multiplying Parts with Parts, that there will not be so many Figures in the Product, as there ought to be places of Decimal Parts by the Rule: In that Case you must supply their Defect by prefixing Cyphers to the Product; as in these Examples.

EXAMPLE 5.

$$\begin{array}{r} ,2365 \\ ,2435 \\ \hline 11825 \\ 7095 \\ 9460 \\ 4730 \\ \hline ,05758775 \end{array}$$

EXAMPLE 6.

$$\begin{array}{r} ,0347 \\ ,0236 \\ \hline 2082 \\ 1041 \\ 694 \\ \hline ,00081892 \end{array}$$

When any proposed Number of Decimals is to be multiplied with 10 : 100 : 1000 : 10000, &c. It is only removing the separating Point in the Multiplicand, so many places towards the Right-hand, as there are Cyphers in the Multiplier.

Thus, $,578 \times 10 = 5,78$. And, $,578 \times 100 = 57,8$.

Again, $,578 \times 1000 = 578$. Or, $,578 \times 10000 = 5780$.

These

These things being considered, it will be easy to multiply Decimals, and determine their true Products. As in these following Examples.

57,056 multiplied into 0,578 will produce 32,978368

7,6543 into 5,4246 will produce 41,52151578

$$0,56879 \times 0,05674 = 0,0322731446$$

$$0,03246 \times 0,02364 = 0,0007672544$$

$$87649 \times 0,03687 = 3231,61863$$

$$94,35786 \times 6,57869 = 620,7511100034$$

$$3,141592 \times 52,7438 = 165,6995001296$$

Now it oftentimes happens, that it will be needless to express all the Figures of the Product at large, (especially, when the Factors have each of them many places of Decimal Parts, as in the two last Examples) only so many of them as may suffice for the intended Design; and yet the Product may be as true to so many Figures as are retained, as if the Factors had been multiplied at large. And such compendious Contractions are not only of Curiosity, but may also be found of great Ease and Use to the ingenious Practitioner; especially in resolving adaffected Equations, or in calculating of Trigonometrical Problems by the Natural Sines and Tangents, &c. All which may be thus performed.

Viz. Set the Unit's Place of the Multiplier directly underneath that Figure of the Multiplicand, whose Place you intend to keep in the Product; and place all the other Figures of the Multiplier in a quite contrary Order to the usual way. Then in multiplying, always begin at that Figure of the Multiplicand which stands over the Figure wherewith you are then a multiplying, setting down the first Figure of each particular Product, directly underneath one another; yet herein you must have a due Regard to the Increase which would arise out of the two next Figures to the Right-hand of that Figure in the Multiplicand which you then begin with.

E X A M P L E.

Let it be required to multiply 3,141592 with 52,7438 and let there be only four Places of Decimal Parts retained in the Product.

If the proposed Numbers were to be multiplied at large they must stand in a direct Order as usual.

Thus, $\left\{ \begin{array}{l} 3,141592 \\ \underline{52,7438} \end{array} \right\}$ And would produce ten Places of Parts, as in the last Example.

*

But

But seeing it is required to have only four Places of those Parts in the Product, set them down as before directed, and they will stand

Thus	{	3,141592	The Multiplicand placed as before.
	{	8347,25	The multiplier in a reverse Order.
		1570796	The Product with 5, regard had to 5 times 2.
		62832	The Product with 2, increased with 9×2 .
		21991	Product with 7, increased with $5 \times 7 + 9 \times 7$.
		1257	Product with 4, increased with $1 \times 4 + 5 \times 4$.
		94	Product with 3, increased with 4×3 .
		25	Product with 8, increased with $4 \times 8 + 1 \times 8$.
		165,6995	The true Product as was required.

The Reason of this Contraction is very obvious from the whole Operation wrought at large.

Thus { 3,141592
52,7438

25	132736
94	24776
1256	6368
21991	144
62831	84
1570796	0
165,6995	001296

From hence it is evident, that all the Figures in the Square to the Right-hand, are wholly omitted in the former Contraction; and that the last single Product here, is the first there; consequently the Reason of placing the Multiplier in a reverse Order, must needs appear very plain.

EXAMPLE 3.

Suppose it were required to multiply 257,356 with 76,48 and to have only the entire Product of integers.

257,356
84,67
18015
1544
103
20
19682

The same at large { 257,356
76,48

20	58848
102	9424
1544	136
18014	92
19682	58688

The chiefest Care and Difficulty that attends these Contractions, is the true setting down of the Units place in the Multiplier underneath the proper Figure of the Multiplicand, according to the designed Product.

Viz.

Viz. In Example 1. It was required to have four Places of Decimal Parts in the Product; therefore the Unit's Place of the Multiplier, was set under the fourth Place of Decimals in the Multiplicand: And in Example 2, because it was required to have an entire Product of Integers only; therefore the Unit's Place of the Multiplier, was set under the Unit's Place of the Multiplicand. This, I say, being once rightly understood, will render the Method easy in Practice.

Sect. 4. Division of Decimals.

DIVISION is accounted the most difficult Part of Decimal Arithmetick: In order therefore to make it plain and easy, it will be convenient to resume what has been said in page 25.

Viz. { *The Quotient Figure is always of the same Value or Degree with that Figure of the Dividend, under which the Unit's Place of it's Product stands.*

As for Instance, Let 294 be divided by 4.

4) 294	{	This is not 7 but 70, because the Units Place of 4×7 stands under the Tens Place of the Dividend.
28		
14	(3	But this is only 3.
12		

Remains (2) Hence $73\frac{2}{4}$ is the Quotient,

Now if to the Remainder 2 there be annexed a Cypher (thus, 2,0) and then divided on, it must needs follow that the Unit's Place of the Product arising from the Divisor into the Quotient, will stand under the annexed Cypher; consequently the Quotient Figure will be of the same Value or Degree with the Place of that Cypher: But that is the next below the Unit's Place, therefore the Quotient Figure is of the next Degree or Place below Unity; That is, in the first Place of Decimal Parts.

Thus 4) 2,0 (,5

So that 4) 294,0 (73,5 the true Quotient required.

This being well understood, Division of Decimals may (in all the various Cases) be easily performed. However, that it may be rendered plain and easy even to the meanest Capacity, if possible; Let Division be again defined, as in page 21.

K

Viz.

Viz. If that Number which divides another, be multiplied with the Number which is quoted, their Product will be the Number divided.

This Definition alone (if compared with the Rule, page 61.) will afford a general Rule for discovering the true Value of the Quotient Figure in Division of Decimals.

Rule { The Place of Decimal Parts in the Divisor and Quotient, being counted together, must always be equal in Number with those in the Dividend. And from this general Rule ariseth four Cases.

Case 1. When the Places of Parts in the Divisor and Dividend are equal, the Quotient will be whole Numbers.

As in these Examples.

$$\begin{array}{r} 8,45) 295,75 \text{ (35)} \\ \underline{253 \ 5} \\ 42 \ 25 \\ \underline{42 \ 25} \\ (0) \end{array}$$

$$\begin{array}{r} 0,0078) ,4368 \text{ (56)} \\ \underline{390} \\ 468 \\ \underline{468} \\ (0) \end{array}$$

Case 2. When the Places of Parts in the Dividend exceed those in the Divisor; cut off the Excess for Decimal Parts in the Quotient. As in these Examples.

$$\begin{array}{r} 24,3) 780,516 \text{ (32,12)} \\ \underline{729} \\ 515 \\ \underline{486} \\ 291 \\ \underline{243} \\ 480 \\ \underline{486} \\ (0) \end{array}$$

$$\begin{array}{r} ,534) ,30438 \text{ (,57)} \\ \underline{2670} \\ 3738 \\ \underline{3738} \\ (0) \end{array}$$

$$\begin{array}{r} 436) 34246,056 \text{ (78,546)} \\ \underline{3052} \\ 3726 \\ \underline{3488} \\ 2380 \\ \underline{2180} \\ 2005 \\ \underline{1744} \\ 2616 \\ \underline{2616} \\ (0) \end{array}$$

Case 3. When there are not so many Places of Parts in the Dividend, as are in the Divisor; annex Cyphers to the Dividend to make them equal. Then will the Quotient be whole Numbers, as in Case 1.

E X-

E X A M P L E S.

Let it be required to divide 192,1 by 7,684, and 441 by ,7875.

$$\begin{array}{r} 7,684 \overline{) 192,100} \quad (25 \\ \underline{153 \ 68} \\ 38 \ 420 \\ \underline{38 \ 420} \\ (0) \end{array}$$

$$\begin{array}{r} ,7875 \overline{) 441,0000} \quad (560 \\ \underline{393 \ 75} \\ 47 \ 250 \\ \underline{47 \ 250} \\ (0) \end{array}$$

Case 4. If after Division is finished, there are not so many Figures in the Quotient, as there ought to be Places of Parts by the general Rule; supply their defect by prefixing Cyphers to it.

E X A M P L E S.

Let it be required to divide 7,25406 by 957.

957) 7,25406 (,00758 the true Quotient required.

$$\begin{array}{r} 6 \ 699 \overline{) 5550} \\ \underline{4785} \\ 7656 \\ \underline{7656} \\ (0) \end{array}$$

Again ,575) ,0007475 (,0013

$$\begin{array}{r} 575 \overline{) 1725} \\ \underline{1725} \\ (0) \end{array}$$

Note, When Decimal Numbers are to be divided by 10. 100. 1000. 10000. &c. that is, when the Divisor is an Unit with Cyphers; Division is performed by removing or placing the separating Point in the Dividend, so many Places towards the Left-hand, as there are Cyphers in the Divisor.

E X A M P L E.

$$\begin{array}{l} 10) 5784 \quad (578,4 \\ 1000) 5784 \quad (5,784 \end{array}$$

$$\begin{array}{l} 100) 578,4 \quad (57,84 \\ 10000) 578,4 \quad (,05784 \end{array}$$

Note, These Operations are the direct Converse to those in page 62.

I presume it needless to give more Examples at large; only I shall insert a few Dividends, and Divisors, with their Quotients, wherein are contained all the Varieties that can happen in Division of Decimals.

$$\begin{array}{l} 574) 493 \ 066 \quad (859 \\ 574) 483,066 \quad (,859 \\ 574) 49,3066 \quad (,0859 \\ 5,74) 4930,66 \quad (859 \end{array}$$

$$\begin{array}{l} 5,74) 49,3066 \quad (8,59 \\ 5,74) 493066,00 \quad (85900 \\ ,0574) 493,0665 \quad (8590 \\ ,0574) ,493066 \quad (8,59 \end{array}$$

K 2

There

There is also a compendious Way of contracting Division, like that of Multiplication, *page 64*, by which much Labour may be saved; especially when the Divisor hath many Places of Decimal Parts in it: And it is thus performed.

Having determined how many Places of whole Numbers there will be in the Quotient, if any at all; or if none, of what Value or Place the first Figure in the Quotient will be: Then omit, or prick off one Figure of the Divisor at each Operation; *viz.* for every Figure you place in the Quotient, prick off one in the Divisor; having a due Regard to the Increase which would arise from the Figure so omitted.

E X A M P L E.

Let it be required to divide 70,23 by 7,9863.

The Work contracted.

$$\begin{array}{r}
 7,9863 \overline{) 70,2300} \quad (8,7938 \\
 \underline{63 \ 8904} \\
 6 \ 3390 \\
 \underline{5 \ 5904} \\
 7492 \\
 \underline{7187} \\
 305 \\
 \underline{239} \\
 66 \\
 \underline{64} \\
 (2)
 \end{array}$$

The same at Length.

$$\begin{array}{r}
 7,9863 \overline{) 70,2300} \quad (8,7938 \\
 \underline{63 \ 8904} \\
 6 \ 3396 \\
 \underline{5 \ 5904} \\
 7491 \\
 \underline{7187} \\
 304 \\
 \underline{239} \\
 64 \\
 \underline{63} \\
 07506
 \end{array}$$

The Work contracted I presume is so obvious (if compared with the same at large) that it is needless to give any farther Explanation of it.

Sect. 5. To Reduce Vulgar Fractions into Decimals, and the contrary.

ANY Vulgar Fraction being given, it may be reduced, or rather changed into Decimal Parts equivalent to it. Thus,

Rule { Annex Cyphers to the Numerator, and then divide it by the Denominator, the Quotient will be the Decimal Parts equivalent to the given Fraction; or at least so near it as may be thought necessary to approach.

E X.

EXAMPLE.

It is required to change or reduce $\frac{3}{4}$ into Decimals.

4) 3,00 (.75 The Decimal Parts required.

That is, $\frac{3}{4} = \frac{75}{100} = .75$.

Again $\frac{1}{2} = .5$; thus 2) 1,0 (.5. And $\frac{1}{4} = .25$; 4) 1,00 (.25.

Suppose it were required to change $\frac{1}{7}$ into Decimals.

7) 4,0000000000 (.5714285714 &c. $= \frac{4}{7}$.

Note, When the last Figure of the Divisor, (that is, the Denominator of the proposed Fraction) happens to be one of these Figures; viz. 1. 3. 7. or 9. (as in the Example) then the Decimal Parts can never be precisely equal to the given Fraction; yet by continuing the Division on, you may bring them to be very near the Truth. As in this Example; Suppose it was required to change $\frac{1}{13}$ into Decimal Parts.

13) 1,0000 (.07692307692307 &c. *ad infinitum*.

$$\begin{array}{r}
 91.. \\
 \hline
 90 \\
 78 \\
 \hline
 120 \\
 117 \\
 \hline
 30 \\
 26 \\
 \hline
 40 \\
 39 \\
 \hline
 10
 \end{array}$$

That is, $0,07692307692307 = \frac{1}{13}$ *ferè*.

And from hence it may be farther observed; that in these imperfect Quotients, the Figures do return again and circulate in the same Order as before: as you may easily perceive they begin to do in the seventh Place of both these last Examples.

&c. As at first.

These being understood, it will be easy to find the Decimal Parts equivalent to any known Part or Parts of Coin, Weights, Measures, Time, &c. If you first reduce the given Parts of Coin, &c. into a Vulgar Fraction, whose Denominator is the Number of those known Parts contained in the Integer, and the given Parts it's Numerator,

Examples in Coin, &c.

1. Let it be required to find the Decimals of 16 s. 6 d. First 16 s. $= \frac{16}{20}$ of one Pound, and 6 d. $= \frac{1}{40}$ of 1 l.

But $\frac{16}{20} + \frac{1}{40} = \frac{33}{40}$. Then 40) 33,000 (.825 the Decimal Parts required: That is, $825 = 16$ s. 6 d.

Again, Suppose it were required to find the Decimals equal to 3 l. 13 s. 4 d.

Here

Here 3*l.* is 3 Integers, and 13*s.* = $\frac{13}{20}$ of 1*l.* and 4*d.* = $\frac{4}{240}$. But $\frac{13}{20} + \frac{4}{240} = \frac{136}{240}$. Then 240) 160,000 (0,666666 $\mathcal{E}$. Hence 3*l.* 13*s.* 4*d.* = 3,666666 $\mathcal{E}$. As was required.

2. What are the Decimals equal to $7\frac{3}{4}$ Inches, one Foot being made the Integer.

First, 7 Inches are $\frac{7}{12}$ of 1 Foot, and $\frac{3}{4}$ of 1 Inch are $\frac{3}{48}$. But $\frac{7}{12} + \frac{3}{48} = \frac{31}{48}$. Then 48) 31,000 (,64583 $\mathcal{E}$. = $7\frac{3}{4}$ Inches.

3. Let it be required to change 8 Oz. 19 Pwt. 8 Grains into Decimals; one Pound Troy being the Integer.

These being reduced into the least Terms, and added together, will become $\frac{3104}{3780}$ of 1 Pound.

Then 3780) 4304,000 (,74722 $\mathcal{E}$. The Decimals required.

And thus may any proposed Parts of Coin, Weights, Measures, $\mathcal{E}$. be reduced or changed into Decimal Parts; which perhaps may at first seem somewhat tedious in Practice, but being a little acquainted with them it will be found very easy; and the ingenious Practitioner will (with a little Consideration) soon find how to reduce them almost mentally; or with the help of a very few Figures, without the use of such large Tables as are usually inserted in Books of Decimal Arithmetick; or at most they may be contracted into such as these following, which if duly applied to those Tables in Chap. 3. will be found very useful.

Decimal Tables.

<i>In English Coin.</i> 0,05..... = 1<i>s.</i> 0,0046667 = 1<i>d.</i> 0,00104167 = 1 <i>Farthing</i> 1<i>l.</i> being the <i>Integer</i>.	<i>Averdupois Weight.</i> 0,0625.... = 1 <i>Ounce</i> 0,00390625 = 1 <i>Dram</i>. 1 lb. being the <i>Integer</i>.
<i>Troy Weight.</i> 0,05..... = 1 <i>Pwt.</i> 0,00208333 = 1 <i>Grain</i>. 1 Oz. being the <i>Integer</i>.	<i>Averdupois Great Weight.</i> 0,25..... = $\frac{1}{4}$ <i>C.</i> 0,00892857 = 1 lb. 0,00055803 = 1 <i>Ounce</i>. 1 <i>C.</i> being the <i>Integer</i>.
<i>Apothecaries Weight.</i> 0,125..... = 1 <i>Dram</i>. 0,04166667 = 1 <i>Scruple</i>. 0,00208333 = 1 <i>Grain</i>. 1 Oz. being the <i>Integer</i>.	<i>Time.</i> 0,04166667 = 1 <i>Hour</i>. 0,00069444 = 1 <i>Minute</i>. 0,00001157 = 1 <i>Second</i>. 1 <i>Day</i>, or 24 <i>Hours</i>, being made the <i>Integer</i>.

The Use of these Tables will be evident by the following
EXAMPLE.

E X A M P L E.

Let it be required to find the Decimal Parts equivalent to 17s. 9d. 2 Farthings

First $0,05 = 1s.$ Therefore $17 \times ,05 = ,85 \dots = 17s.$

And, $,004166 = 1d.$ Therefore, $,004166 \times 9 = ,037494 = 9d.$

Also $2) ,004166 (= ,002083 = \frac{1}{2}d.$

Consequently their Sum, viz. $0,889577 = 17s. 9\frac{1}{2}d.$

Now to find the Value of Decimals in known Parts of Coin or Weights, &c. is only the Converse of the former Work, and is thus performed.

Multiply the given Decimals with the Denominator of the Vulgar Fraction required: That is, multiply the Decimals with such a Number of Units as are contained in the next lower Denomination of that Kind or Species which your Decimal is of; and the Product will be the Number required.

E X A M P L E.

1. What is the Value of 0,825 Decimals of 1 Pound Sterling. That is, how many Shillings, Pence, &c. $= ,825$. First, the next lower Denomination is 20, because 20s. make one Pound.

Therefore 0,825

20

Shillings $\frac{16,500}{12}$ and Parts of 1 Shilling,

12

Pence $\frac{6,000}{12}$ Answer $0,825 = 16s. 6d.$

Again, What are the known Parts of English Coin equal to 3,666666 Decimals.

Here the 3 Integers are 3 Pounds. Then ,666666

20

Shillings $\frac{13,33320}{12}$

12

Answer $3,666666 = 3l. 13s. 4d.$

666640

3 3332

Pence $\frac{3,999840}{12} = 4 \text{ near.}$

What is the Value of 0,74722 Parts of 1lb Troy.

First, ,74722

12

1 49444

7 4722

Oz. $\frac{8,96664}{12}$

Then, ,96664

20

Pwts. $\frac{19,33280}{24}$

Again, ,33280

24

1 3312

6 656

$\frac{7,98720}{24}$

These collected are Oz. 8. Pwr. 19. Gr. 8. very near.

And

And thus any proposed Number of Decimals may be turned or changed into the known Parts of what they represent, *viz.* Whether they be Parts of Coin, Weights, Measures, or Time, &c.

I have omitted inserting more Examples of this kind, because I take the Excellency, and indeed the chief Use of Decimal Fractions to consist more in Geometrical Computations, than in the common or practical Parts of Arithmetick, as will appear further on; although even in those they are very useful upon several Accounts; especially in the Computations of Interest and Annuities, &c. But of that more in it's proper Place. I shall therefore conclude this Chapter, with a Remark or two upon the Nature and Properties of Fractions in general.

If any given Number (whether it be whole or mixed) be multiplied with a Fraction either Vulgar or Decimal, the Product will be less than the Multiplicand, in such a Proportion as the multiplying Fraction is less than an Unit or 1.

That is; as the Denominator of the Fraction is to it's Numerator, so will the given Number be to the Product.

Therefore, whenever any Number is to be multiplied with a Fraction, whose Numerator is an Unit: Divide that Number by the Denominator of the Fraction, and the Quotient will be the Product required. Thus $12 \times \frac{1}{4} = 3$. And $12 \div 4 = 3$. Again, $12 \times \frac{1}{2} = 6$. And $12 \div 2 = 6$, &c.

From hence it follows, that if any Number be divided by a Fraction, the Quotient will be greater than the Dividend, by such a Proportion as Unity is greater than the dividing Fraction.

Thus $12 \div \frac{1}{4} = 48$, *viz.* $\frac{1}{4} : 1 :: 12 : 48$, &c. But the Truth of these will be best understood after the next Chapter.

C H A P. VI.

Of Continued Proportions, and how to change or vary the Order of Things.

Sect. 1. *Concerning Arithmetical Progression, usually called Arithmetical Proportion Continued.*

WHEN any Rank or Series of Numbers do either increase or decrease by an equal Interval or common Difference; those Numbers are said to be in Arithmetical Progression.

As

As $\left\{ \begin{array}{l} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \text{ \&c.} \\ 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 \end{array} \right\}$ Here the Interval or common Difference is 1.

Or $\left\{ \begin{array}{l} 2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot \text{\&c.} \\ 1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot \text{\&c.} \end{array} \right\}$ Here the common Difference is 2.

And so of any other Series, whose common Difference is $3 \cdot 4 \cdot 5 \cdot \text{\&c.}$

Lemma 1.

If any three Numbers be in Arithmetical Progression, the Sum of the two Extremes (*viz.* the first and last) will be equal to the Double of the Mean or middle Number.

As in these, $2 \cdot 4 \cdot 6$. Or $3 \cdot 6 \cdot 9$. Or $3 \cdot 7 \cdot 11$.

Viz. $2 + 6 = 4 + 4$. Or $3 + 9 = 6 + 6$. And $3 + 11 = 7 + 7 \cdot \text{\&c.}$

Lemma 2.

If any four Numbers are in Arithmetical Progression, the Sum of the two Extremes will be equal to the Sum of the two Means.

As in these, $2 \cdot 4 \cdot 6 \cdot 8$. Or $3 \cdot 6 \cdot 9 \cdot 12$.

Viz. $2 + 8 = 4 + 6$. And $3 + 12 = 6 + 9 \cdot \text{\&c.}$

Corollary 1.

From these two Lemmas it is easy to conceive, that if never so many Numbers be in Arithmetical Progression, the Sum of the two Extremes will be equal to the Sum of any two Means, that are equally distant from those Extremes.

As in these, $2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16$.

Then $2 + 16 = 4 + 14 = 6 + 12 = 8 + 10$.

Or if the Number of Terms be odd, as these,

$2 \cdot 4 \cdot 6 \cdot 8 \cdot 10 \cdot 12 \cdot 14 \cdot 16 \cdot 18 \cdot \text{\&c.}$

Then $2 + 18 = 4 + 16 = 6 + 14 = 8 + 12 = 10 + 10$.

Lemma 3.

Every Series of Numbers in Arithmetical Progression is composed of the Interval or common Difference, so often repeated as there are Terms in the Progression, except the first.

As in these, $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17 \cdot \text{\&c.}$

Here the Interval or common Difference being two, it will be $1 + 2 = 3$. $3 + 2 = 5$. $5 + 2 = 7$. $7 + 2 = 9$. $9 + 2 = 11$. $11 + 2 = 13$. $13 + 2 = 15$. $15 + 2 = 17 \cdot \text{\&c.}$

Corollary 2.

Hence it is evident, that the Difference betwixt the two Extremes (*viz.* 1 and 17) is composed of the common Difference, multiplied into the Number of all the Terms, excepting the first.

As in the aforesaid Progression, $1 \cdot 3 \cdot 5 \cdot 7 \cdot 9 \cdot 11 \cdot 13 \cdot 15 \cdot 17$.

L

The

The Number of Terms without the first is 8 } Multiply
 The common Difference is 2 }
 The Difference betwixt the two Extremes is 16

Proposition 1.

In any Series of Numbers in Arithmetical Progression, the two Extremes, and the Number of Terms being given, thence to find the Sum of all the Series.

Theorem 1. $\left\{ \begin{array}{l} \text{Multiply the Sum of the two Extremes into the} \\ \text{Number of all the Terms; and divide the Product} \\ \text{by 2. The Quotient will be the Sum of all that Series.} \end{array} \right.$
Per Corol. 1.

E X A M P L E 1.

It is required to find the Number of all the Strokes a Clock strikes in one whole Revolution of the Index, viz. twelve Hours.

Here $1 + 12 = 13$ the Sum of the two Extremes.
 $\times 12$ the Number of all the Terms
 26
 13

Then 2) 156 (78. The Number of Strokes required

E X A M P L E 2.

Suppose one Hundred Eggs were placed in a Right Line a Yard distant from one another, and the first Egg were a Yard from a Basket; whether or no may a Man gather up these 100 Eggs singly one after another, still returning with every Egg to the Basket and putting it in, before another Man can run four Miles. That is, which will run the greater Number of Yards.

In this Question $200 + 2 = 202$ Is the Sum of the two Extremes.
 And $\times 100$ Is the Number of all the Terms.

Then 2) 20200 (10100 } The Number of Yards
 he runs that takes up
 the Eggs.

Now 4 Miles = 7040 Yards } The Yards he runs that takes up
 But $10100 - 7040 = 3060$ } the Eggs more than the other.

Proposition 2.

In any Series of Numbers in Arithmetical Progression, the two Extremes and Number of Terms being given; thence to find the common Difference of all the Terms in that Series.

Theorem 2. $\left\{ \begin{array}{l} \text{The Difference betwixt the two Extremes, being} \\ \text{divided by the Number of Terms less an Unit or 1.} \\ \text{The Quotient will be the common Difference of the} \\ \text{Series. Per Corol. 2.} \end{array} \right.$

E X A M P L E

E X A M P L E 1.

One had Twelve Children that differed alike in all their Ages; the youngest was Nine Years old, the eldest was Thirty-six and a half; what was the Difference of their Ages, and the Age of each?

Here $36,5 - 9 = 27,5$ The Difference of the two Extremes.

And $12 - 1 = 11$. The Numbers of Terms less an Unit.

Then 11) $27,5$ (2,5 The common Difference required.

Consequently $9 + 2,5 = 11,5$ The Age of the youngest but one.

And $11,5 + 2,5 = 14$ The Age of the youngest but two. And so on for the rest. *Per Corol. 2.*

E X A M P L E 2.

A Debt is to be discharged at eleven several Payments to be made in Arithmetical Progression. The first Payment to be Twelve Pounds Ten Shillings, and the last to be Sixty-three Pounds. What is the whole Debt, and what must each Payment be?

Per Theorem 1. Find the whole Debt thus:

$12,5 + 63 = 75,5$ The Sum of the Extremes.

11 The Number of Terms.

$$\begin{array}{r} 75\ 5 \\ 11 \\ \hline 830\ 5 \end{array}$$

2) $830,5$ ($415,25 = 415\text{ l. } 5\text{ s.}$ The whole Debt.

Then, *per Theorem 2.* find the common Difference of each Payment.

Thus $63 - 12,5 = 50,5$ The Difference of the Extremes.

And $11 - 1 = 10$ The Number of Terms less 1.

Then 10) $50,5$ ($5,05 = 5\text{ l. } 1\text{ s.}$ The common Difference.

l. s. l. s. l. s.

Consequently $12\text{ l. } 10\text{ s.} + 5\text{ l. } 1\text{ s.} = 17\text{ l. } 11\text{ s.}$ The second Payment.

l. s. l. s. l. s.

And $17\text{ l. } 11\text{ s.} + 5\text{ l. } 1\text{ s.} = 22\text{ l. } 12\text{ s.}$ The third Payment, &c.

E X A M P L E 3.

A Man is to travel from *London* to a certain Place in ten Days, and to go but two Miles the first Day, increasing every Day's Journey by an equal Excess; so that the last Day's Journey may be Twenty-nine Miles; what will each Day's Journey be, and how many Miles is the Place he goes to distant from *London*?

L 2

First

First $29 - 2 = 27$ The Difference of the Extremes.

And $10 - 1 = 9$ The Number of Terms less 1.

Then $9 \times 27 \div 3$ The common Difference.

Consequently $2 + 3 = 5$ The second Day's Journey.

And $5 + 3 = 8$ The third Day's Journey, &c.

Again $29 + 2 = 31$ The Sum of the Extremes.

10 The Number of Terms.

2) 310 (155 The Distance required.

There are eighteen Theorems more relating to Questions in Arithmetical Progression; but because they would require a great many Words to shew the Reason of them: I therefore refer the Reader to the Second Part, *viz.* That of *Algebra*, where he may find their *Analytical Investigation*.

SECT. 2. Concerning Geometrical Proportion continued;
sometimes called Geometrical Progression.

WHEN a Rank or Series of Numbers do either increase by one common Multiplier, or decrease by one common Divisor; Those Numbers are said to be in Geometrical Proportion continued.

As { $2 \cdot 4 \cdot 8 \cdot 16 \cdot 32$. &c. here 2 is the common Multiplier.
 $64 \cdot 32 \cdot 16 \cdot 8 \cdot 4$. &c. here 2 is the common Divisor.

Or { $2 \cdot 6 \cdot 18 \cdot 54 \cdot 162$. &c. here 3 is the common Multiplier.
 $162 \cdot 54 \cdot 18 \cdot 6 \cdot 2$. here 3 is the common Divisor.

Note, The common Multiplier (or Divisor) is called the Ratio; and it shews the Habitude or Relation the Numbers have to one another, *viz.* whether they are Double, Triple, Quadruple, &c. which *Euclid* thus defines.

Ratio (or Rate) is the mutual Habitude or Respect of two Magnitudes (consequently two Numbers) of the same kind each to other, according to Quantity, *Eucl.* 5. Def. 3.

Proportion (rather Proportionality) is a Similitude of Ratio's. *Eucl.* 5. Def. 4.

So that there cannot be less than three Terms to form a Proportionality or Similitude of Ratio's; and if but three Terms, the second must supply the Place of two, As in these $2 \cdot 4 \cdot 8$. That is, $2 : 4 :: 4 : 8$. (of :: see page 5.)

Here 4 the middle Term supplies the Place of two Terms, to wit, of the second and third; 8 bearing the same Reason,
Liken es,

Likeness, or Proportion, to 4, as 4 doth to 2. *viz.* As 2 : is to 4 :: So is 4 : to 8.

Lemma 1.

If three Numbers are proportional, the Rectangle, or Product of the two Extremes; *viz.* of the first and last Terms will be equal to the Square of the Mean or middle Term. (20 *Eucl.* 7.)

As in these 2 : 4 :: 4 : 8. Here $8 \times 2 = 16$ the Product of the Extremes.

And $4 \times 4 = 16$ the Square of the Mean. *Ergo* $8 \times 2 = 4 \times 4$.

Corol. 1.

Hence it follows, that if the Product of any two Numbers be equal to the Square of a third Number; those three Numbers will be in Proportion.

Lemma 2.

If four Numbers are proportional, the Product of the two Extremes will be equal to the Product of the two Means, (19 *Euclid* 7.)

As in these, 2 : 4 :: 8 : 16. Here $16 \times 2 = 32$.

And $8 \times 4 = 32$. Consequently $16 \times 2 = 8 \times 4$.

Corol. 2.

From hence it follows, that if the Product of any two Numbers, be equal to the Product of any other two Numbers, those four Numbers are Proportionals.

And from these two *Lemma's* it will be easy to conceive, that if never so many Numbers are in continued Proportion; the Product of the two Extremes, will be equal to the Product of any two Means, that are equally distant from the Extremes.

As in these 2 . 4 . 8 . 16 . 32 . 64. &c.

Here $64 \times 2 = 32 \times 4 = 16 \times 8$. &c. And if the Number of Terms be odd,

As in these 2 . 4 . 8 . 16 . 32 . 64 . 128. &c.

Then $128 \times 2 = 64 \times 4 = 32 \times 8 = 16 \times 16$.

Note, The Character made Use of to signify continued Proportionals is $\div\div$.

In

In every Series of $\div\div$ (*viz. of continued Proportionals*) that Number which is compared to another, is called the Antecedent of the Ratio; and that Number to which it is compared, is called it's Consequent.

As in these, $2 : 4 :: 4 : 8$. Here 2 is the Antecedent, and 4 is the Consequent; and 4 the middle Term is an Antecedent to 8 it's Consequent: whence it follows, that in every Series of $\div\div$ all the middle Terms between the first and last are both Antecedents and Consequents.

As in these, $2 . 4 . 8 . 16 . 32 . 64 . \&c.$ Here $4 . 8 . 16 . 32 .$ are both Consequents and Antecedents.

For $2 : 4 :: 4 : 8 :: 8 : 16 :: 16 : 32 :: 32 : 64 . \&c.$

So that all the Terms except the last are Antecedents. And all the Terms except the first are Consequents.

Lemma 3.

If never so many Numbers are proportional, it will be: As any one of the Antecedents is to it's Consequent: So will the Sum of all the Antecedents be; to the Sum of all the Consequents. (*12 Euclid 5.*)

That is, in the foregoing Series.

$$2 : 4 :: 2 + 4 + 8 + 16 + 32 : 4 + 8 + 16 + 32 + 64.$$

For it is evident, that $4 + 8 + 16 + 32 + 64$, the Sum of all the Consequents, is double to $2 + 4 + 8 + 16 + 32$ the Sum of all the Antecedents; as 4 is to 2, according to the Ratio, and would have been Triple, or Quadruple, $\&c.$ had the Ratio been 3 or 4, $\&c.$

Note, In every Series of $\div\div$ the Ratio is found by dividing any of the Consequents by it's Antecedent.

As in these $2 : 6 :: 6 : 18 :: 18 : 54 :: 54 : 162.$

Here 2) 6 (3 the Ratio. Or 6) 18 (3 $\&c.$

From the second and third Lemma's may be raised two general Theorems or Rules, for finding the Sum of any Series in $\div\div$ without a continued Addition of all the Terms.

Let the Series $2 . 4 . 8 . 16 . 32 . 64 . 128.$ be given, to find it's Sum.

Suppose $z =$ the Sum of all the Terms.

Then will $z - 128 =$ the Sum of all the Antecedents.

And $z - 2 =$ the Sum of all the Consequents.

But $2 : 4 :: z - 128 : z - 2.$ per Lemma 3.

Ergo $4z - 512 = 2z - 4.$ per Lemma 2.

Consequently

Consequently $4z - 2z = 512 - 4$.

Theorem. $z = \frac{512 - 4}{4 - 2}$ In Words at length thus,

Theorem 1. $\left\{ \begin{array}{l} \text{From the Product of the second and last Terms} \\ \text{subtract the Square of the first Term, and that Re-} \\ \text{mainder being divided by the second Term less the} \\ \text{first, will give the Sum of all the Series.} \end{array} \right.$

Or if the first Term, the common Ratio, and the last Term be only given. Then,

Theorem 2. $\left\{ \begin{array}{l} \text{Multiply the last Term into the Ratio, and from} \\ \text{their Product subtract the first Term; divide that} \\ \text{Remainder by the Ratio less Unity or 1, and it will} \\ \text{give the Sum of all the Series.} \end{array} \right.$

For $4z - 2z = 512 - 4$. As above.

Consequently $2z - z = 256 - 2$. viz. the last divided by 2.

Then $z = \frac{256 - 2}{2 - 1}$ Theorem 2.

E X A M P L E.

Let 2 . 6 . 18 . 54 . 162 . 486. be the given Series. Here 2 is the first Term, 3 is the Ratio, and 486 the last Term.

But $486 \times 3 = 1458$. And $1458 - 2 = 1456$.

Then $3 - 1 = 2$) 1456 (728 the Sum required.

That is, $728 = 2 + 6 + 18 + 54 + 162 + 486$.

Since in either of these Theorems it is required to have the last Term known (the which in a long Series of $\div$, will be very tedious to come at by a continued Multiplication) it will therefore be convenient to shew how to obtain either the last Term or any other Term, whose Place is assigned, without producing all the Terms.

In order to that, it will be necessary to premise the Coherence or Similitude that is betwixt Numbers in Arithmetical Progression and those in Geometrical Proportion.

If to any Series of Numbers in $\div$, when the first Term is not an Unit or 1, there be assigned a Series of Numbers in Arithmetical Progression, beginning with an Unit or 1, and whose common Difference is 1. called Indices or Exponents.

Thus, $\left\{ \begin{array}{l} 1 . 2 . 3 . 4 . 5 . 6 . 7. \text{ Indices} \\ 2 . 4 . 8 . 16 . 32 . 64 . 128. \&c. \div \end{array} \right.$

Then

Then will the Addition or Subtraction of any two of those Indices (or Numbers in Arithmetical Progression) directly correspond with the Product, or Quotient of their respective Terms in the Series of $\div$.

That is, $\left\{ \begin{array}{l} \text{As } 3 + 4 = 7. \\ \text{So } 8 \times 16 = 128 \text{ the seventh Term in } \div \end{array} \right.$

Again, $\left\{ \begin{array}{l} \text{As } 6 + 4 = 10. \\ \text{So } 64 \times 16 = 1024. \text{ the tenth Term in } \div \end{array} \right.$

Or, $\left\{ \begin{array}{l} \text{As } 7 - 3 = 4. \\ \text{So } 128 \div 8 = 16. \end{array} \right.$ Or, $\left\{ \begin{array}{l} \text{As } 6 - 2 = 4. \\ \text{So } 64 \div 4 = 16. \end{array} \right. \&c.$

But if the Series of $\div$ begin with an Unit, the Indices must begin with a Cypher.

As in these, $\left\{ \begin{array}{l} 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6. \&c. \\ 1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64. \end{array} \right.$

Now by the help of the Indices, and a few of the first Terms in any Series of $\div$; it is plain that any Term whose Place or Distance from the first Term is assigned, may be speedily obtained without producing the whole Series.

E X A M P L E 1. •

A Man bought a Horse, and was to give a Farthing for the first Nail, two for the second, four for the third, $\&c.$ in $\div$, the Number of Nails was to be 7 in every Shoe, *viz.* 28 Nails in all. What must he have paid for the Horse?

First, $\left\{ \begin{array}{l} 0 \cdot 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5. \text{ Indices} \\ 1 \cdot 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32. \text{ Farthings in } \div \end{array} \right.$

Then, $\left\{ \begin{array}{l} 5 + 5 = 10 \\ 32 \times 32 = 1024 \end{array} \right.$ And, $\left\{ \begin{array}{l} 10 + 10 = 20 \\ 1024 \times 1024 = 1048576 \end{array} \right.$

Again, $\left\{ \begin{array}{l} 4 + 3 = 7 \\ 16 \times 8 = 128 \end{array} \right.$ Lastly, $\left\{ \begin{array}{l} 20 + 7 = 27 \\ 1048576 \times 128 = 134217728 \end{array} \right.$

Which is here to be accounted the 28 and last Term. Because the first Term in the Series is 1, which doth neither multiply nor divide.

Now this 134217728 being the Number of Farthings to be paid for the last Nail, by it the common Ratio which is 2, and the first Term which is 1, may be found the Sum of all the Series, *per Theorem 2.*

134217728

2

268435456 From this Product subtract 1.

Viz. 268435456 — 1 = 268435455. Then 2 — 1 = 1 the Divisor.

Consequently 268435455 is the Sum of all the Series, or Price of the Horse in Farthings, which being brought into Pounds, (See page 46) will be 279620*l.* 5*s.* 3*d.* 3*qrs.*

E X A M P L E 2.

A cunning Servant agreed with a Master (unskilled in Numbers) to serve him Eleven Years without any other Reward for his Service but the Produce of one Wheat Corn for the first Year; and that Product to be sowed the second Year, and so on from Year to Year until the end of the Time, allowing the Increase to be but in a ten-fold Proportion.

It is required to find the Sum of the whole Produce.

First { 1 . 2 . 3 . 4 . 5. Indices or Years.
10 . 100 . 1000 . 10000 . 100000 Wheat Corns in ÷÷

Then { As 4 + 2 = 6.
So 10000 × 100 = 1000000. the 6th Year's Produce.

And { 6 + 5 = 11.
1000000 × 100000 = 100000000000. The eleventh or last Year's Produce.

Then (either by *Theorem* 1, or 2) the Sum of all the Series will be 111111111110 Corns. Now it may be computed from Page 31 and 34, that 7680 Wheat Corns, round and dry out of the middle of the Ear, will fill a Statute Pint. If so,

Then 7680) 111111111110 (14467592 Pints, but 64 Pints are contained in a Bushel.

Therefore 64) 14467592 (226056 $\frac{1}{8}$ Bushels. Suppose it to be fold for 3 Shillings the Bushel;

Then { 226056 $\frac{1}{8}$
3

Shillings - 678168 $\frac{3}{8}$ = 33908*l.* 8*s.* 4 $\frac{1}{2}$ *d.* A very good Recompence for Eleven Years Service.

There are several pretty Questions resolved by Numbers in Arithmetical Progression, and by those in ÷÷, which the ingenious Learner will easily perceive hereafter; viz. When we come to the Solution of Questions relating to Interest and Annuities, &c.

M

There

There is also a third Kind of Proportion, called Musical, which being but of little or no common Use, I shall therefore give but a short Account of it.

Musical Proportion or Habitude is, when of three Numbers; the first hath the same Proportion to the third, as the Difference between the first and second hath to the Difference between the second and third.

As in these, 6 . 8 . 12. viz. $6 : 12 :: 8 - 6 : 12 - 8$

If there are four Numbers in Musical Proportion; The first will have the same Proportion to the fourth, as the Difference between the first and second hath to the Difference between the third and fourth.

As in these 8 . 14 . 21 . 84.

Here $8 : 84 :: 14 - 8 = 6 : 84 - 21 = 63$.

That is, $8 : 84 :: 6 : 63$.

The Method of finding out Numbers in Musical Proportion, is best expressed by Letters; as shall be shewed in the *Algebraick Part*.

Sect. 3. *How to Change or Vary the Order of Things, &c.*

THIS being a Thing not treated of in any common Books of Arithmetick (that I have had the Opportunity of perusing), made me think it would be acceptable to the young Learner, to know how oft it is possible to vary or change the Order or Position of any proposed Number of Things.

As how many several Changes may be rung upon any proposed Number of Bells; or how many several Variations may be made of any determined Number of Letters, or any other Things proposed to be varied.

The Method of finding out the Number of Changes is by a continual Multiplication of all the Terms in a Series of Arithmetical Progressions, whose first Term and common Difference is Unity or 1. And the last Term the Number of Things proposed to be varied, viz. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7$, &c. As will appear from what follows.

1. If the Things proposed to be varied are only two, they admit of a double Position (as to Order of Place) and no more.

$$\text{Thus, } \left\{ \begin{array}{cc} 1 & 2 \\ 2 & 1 \end{array} \right\} = 2 = 1 \times 2.$$

2. And if three Things are proposed to be varied, they may
* be

be changed fix several Ways (as to their Order of Place) and no more.

For, beginning with 1, there will be $\begin{cases} 1 \cdot 2 \cdot 3 \\ 1 \cdot 3 \cdot 2 \end{cases}$

Next, beginning with 2, there will be $\begin{cases} 2 \cdot 1 \cdot 3 \\ 2 \cdot 3 \cdot 1 \end{cases}$

Again, beginning with 3, it will be $\begin{cases} 3 \cdot 1 \cdot 2 \\ 3 \cdot 2 \cdot 1 \end{cases}$

Which in all make 6 or 3 Times 2, viz. $1 \times 2 \times 3 = 6$

Suppose four Things are proposed to be varied;
Then they will admit of 24 several Changes, as to their Order of different Places.

For beginning the Order with 1 it will be $\begin{cases} 1 \cdot 2 \cdot 3 \cdot 4 \\ 1 \cdot 2 \cdot 4 \cdot 3 \\ 1 \cdot 3 \cdot 2 \cdot 4 \\ 1 \cdot 3 \cdot 4 \cdot 2 \\ 1 \cdot 4 \cdot 2 \cdot 3 \\ 1 \cdot 4 \cdot 3 \cdot 2 \end{cases}$

Here is six different Changes.

And for the same Reason there will be 6 different Changes, when 2 begins the Order, and as many when 3 and 4 begins the Order; which in all is $24 = 1 \times 2 \times 3 \times 4$. And by this Method of proceeding, it may be made evident, that 5 Things admit of 120 several Variations or Changes; and 6 Things of 720, &c. As in this following Table.

<i>The Number of Things proposed to be varied.</i>	<i>The manner how their several Variations are produced.</i>	<i>The different Changes or Variations every one of the proposed Numbers can admit of.</i>
1	1	= 1
2	1×2	= 2
3	2×3	= 6
4	6×4	= 24
5	24×5	= 120
6	120×6	= 720
7	720×7	= 5040
8	5040×8	= 40320
9	40320×9	= 362880
10	362880×10	= 3628800
11	3628800×11	= 39916800
12	39916800×12	= 479001600
&c.	&c.	&c.

These may be thus continued on to any assigned Number. Suppose to 24 the Number of Letters in the Alphabet, which will admit of 620448401733239439360000 several Variations.

From these Computations may be started several pretty, and indeed, very strange Questions.

EXAMPLES.

Six Gentlemen, that were travelling, met together by Chance at a certain Inn upon the Road, where they were so pleased with their Host, and each other's Company, that in a Frolick they made a Contract to stay at that Place, so long as they, together with their Host, could sit every Day in a different Order or Position at Dinner; which by the foregoing Computations will be found near 14 Years. For they being made 7 with their Host, will admit of 5040 different Positions; but 5040 being divided by $365\frac{1}{4}$ (the Number of the Days in one Year) will give 13 Years and 291 Days. A very pretty Frolick indeed.

I have been told, that before the Fire of *London* (which happened *Anno* 1666) there were 12 Bells in *St Mary Le Bow's Church* in *Cheapside, London*. Suppose it were required to tell how many several Changes might have been rung upon those 12 Bells; and at a moderate Computation how long all those Changes would have been ringing but once over.

First, $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 = 479001600$, the Number of Changes.

Then supposing there might be rung 10 Changes in one Minute: viz. $12 \times 10 = 120$ Strokes in a Minute, which is 2 Strokes in a Second of Time: Now according to that Rate there must be allowed 47900160 Minutes to ring them once over in all their different Changes; viz. 10) 479001600 (47900160.

In one Year there is 365 Days, 5 Hours, and 49 Minutes; which, being reduced into Minutes, is 525949.

Then 525949) 47900160 (91 Years and 26 Days.

So long would those 12 Bells have been continually ringing without any Intermiſſion, before all their different Changes could have been truly rung but once over. It is strange, and seems almost incredible, that a few Things should produce such Varieties.

But that which seems yet more strange and surprising (yea, even impossible to those who are not versed in the Power of Numbers) is,

is, that if two Bells more had been added to the aforesaid 12, they would have advanced the Number of Changes (and consequently the Time) beyond common Belief. For 14 Bells would require (at the same rate of ringing as before) about 16575 Years to ring all their different Changes but once over.

And if it were possible to ring 24 Bells in Changes (and at the same rate of 10 Changes in a Minute, which is 2 Strokes in one Second) they would require more than 1170000000000000000 Years to ring them but once over in all their different Changes; as may easily be computed from the precedent Table.

C H A P. VII.

Of Proportion Disjunct; commonly called the Golden Rule.

Proportion Disjunct or the **Golden Rule**, is either Direct or Reciprocal, called Inverse. And those are both Simple and Compound.

S E C T. I.

Direct Proportion is, when of four Numbers, the first bearing the same Ratio or Proportion to the second; as the third doth to the fourth.

As in these $2 : 8 :: 6 : 24$.

Consequently, the greater the second Term is, in respect to the first; the greater will the fourth Term be, in respect to the third.

That is, as 8 the second Term is 4 Times greater than 2 the first Term: So is 24 the fourth Term, 4 Times greater than 6 the third Term.

Whence it follows, that if four Numbers are in Direct Proportion, the Product of the two Extremes will always be equal to the Product of the two Means, as well in Disjunct as in continued Proportion; according to *Lemma 2. page 77*.

For As $2 : 2 \times 4 :: 6 : 6 \times 4$. Or As $3 : 3 \times 5 :: 6 : 6 \times 5$.
But $2 \times 6 \times 4 = 2 \times 4 \times 6$. Or $3 \times 6 \times 5 = 3 \times 5 \times 6$.

That is, the Product of the Extremes is equal to that of the Means.

Again,

Again, the less the second Term is, in respect to the first; the less will the fourth Term be in respect to the third.

As in these $18 : 6 :: 12 : 4$.

That is, $18 : 18 \div 3 :: 12 : 12 \div 3$.

But $18 \times 12 \div 3 = 18 \div 3 \times 12$. *Viz.* $18 \times 4 = 6 \times 12$.

Consequently $2 . 8 . 6 . 24$. And $18 . 6 . 12 . 4$. are true Proportionals, *per Corol. 2. page 77*.

From these Considerations, comes the Invention of finding a fourth Number in Proportion to any three given Numbers. *Whence it is called the Rule of Three.*

For if the second Number multiplied into the third, be equal to the first multiplied into the fourth, it is easy to conceive, that if the Product of the second and third be divided by the first, the Quotient must needs be the fourth Number. For if that Number, which divides another, be multiplied into the Quotient produced by that Division; their Product will be equal to the Number divided. *See page 21.*

As in these $2 : 8 :: 6 : 24$. Here $8 \times 6 = 48 = 24 \times 2$.

But if $24 \times 2 = 48$, then will $48 \div 2 = 24$. Or $48 \div 24 = 2$.

Note, Any four Numbers in direct Proportion may be varied several Ways. As in these.

Viz. If $2 : 8 :: 6 : 24$. Then $2 : 6 :: 8 : 24$.

And $6 : 24 :: 2 : 8$. Or $24 : 6 :: 8 : 2$. &c.

These Variations being well understood, will be of no small Use in the stating of any Question in this Rule of Three.

When three Numbers are given, and it is required to find a fourth Proportional; the greatest Difficulty (if there be any) will be in the right stating the Question, or abstracting the Numbers out of the Words in the Question, and placing them down in their proper Order.

Now this will be very easy, if it be truly considered, that always two of the three given Terms, are only supposed, and assigned or limit the Ratio or Proportion. The third moves the Question; and the fourth gives the Answer.

As for instance; if 3 Yards of Cloth cost 9 Shillings: What will 6 Yards cost at the same Rate or Proportion?

Here 3 Yards, and 9 Shillings, are two supposed Numbers that imply the Rate; as appears by the Word [if] *viz.* If 3 Yards cost 9 Shillings (then comes the Question) What will 6 Yards cost?

N. B.

N. B. The Term, which moves the Question, hath generally some of those Words before it; viz. **What will? How many? How long? How far? How much?** &c.

Then (carefully observe this; viz.) The first Term in the Supposition must always be of the same kind and Denomination with that Term which moves the Question. And the Term sought will always be of the same kind and Denomination with the second Term in the Supposition.

Thus, $\left\{ \begin{array}{l} \text{yds shil.} \quad \text{yds shil.} \\ 3 : 9 :: 6 : \text{---} \end{array} \right.$

Then

All Questions in direct Proportion may be answered by three several *Theorems*.

Theorem 1. $\left\{ \begin{array}{l} \text{Multiply the second and third Terms together, and} \\ \text{divide their Product by the first Term; the Quo-} \\ \text{tient will be the Answer required.} \end{array} \right.$

$\text{yds shil.} \quad \text{yds shil.}$

Thus $3 : 9 :: 6 : 18.$ The Answer.

3) 54 (18 Shillings, $\left\{ \begin{array}{l} \text{because the second Term} \\ \text{was Shillings.} \end{array} \right.$

Theorem 2. $\left\{ \begin{array}{l} \text{Divide the second Term by the first, then multiply} \\ \text{the Quotient into the third Term; and their Pro-} \\ \text{duct will be the Answer required.} \end{array} \right.$

$\text{yds shil.} \quad \text{yds shil.}$

$3 : 9 :: 6 : 18.$

Thus 3) 9 (=3. Then $3 \times 6 = 18$, as before.

Theorem 3. $\left\{ \begin{array}{l} \text{Divide the third Term by the first, then multiply} \\ \text{the Quotient into the second Term, and their Pro-} \\ \text{duct will be the Answer.} \end{array} \right.$

$\text{yds shil.} \quad \text{yds shil.}$

$3 : 9 :: 6 : 18.$

Thus 3) 6 (=2. And $9 \times 2 = 18$, as before.

Here you see that all the three *Theorems* are equally true; but the first is most general, and usually practised. Yet the two last may be readily performed, when either the second or third Term can be divided by the first; and will be found of singular Use in the *Rules of Fellowship*, &c. as will appear further on.

Quest.

Quest. 2. If 8 Pounds of Tobacco cost 14 Shillings; what will half a hundred Weight (*viz.* 56 Pounds) cost at the same Rate?

Thus 8 lb : 14 s. :: 56 lb : 4l. 18 s. The Answer.

$$\begin{array}{r} 14 \\ \hline 224 \\ 56 \\ \hline 8) 784 (= 98 s. = 4l. 18 s. \end{array}$$

Or thus 8) 56 (= 7. Then $14 \times 7 = 98 s.$ as before.

Quest. 3. If 14 Shillings will buy 8 Pounds of Tobacco; how much will 4l. 18 s. buy after the same Rate?

Stated thus, 14 : 8 lb :: 4l. 18 s. = 98 s. : —

Then $98 \times 8 = 784.$ And 14) 784 (56 lb. The Answer.

Quest. 4. If half a hundred Weight of Tobacco be worth 4l. 18 s. How much may I buy for 14 Shillings at that Rate?

Stated thus, 4l. 18 s. = 98 s. : 56 lb :: 14 s. : —

Then $56 \times 14 = 784.$ And 98) 784 (8 lb. The Answer.

Quest. 5. Suppose 4l. 18 s. will buy 56 Pounds of Tobacco; what will 8 Pounds of the same Tobacco cost?

This Question is thus stated, 56 lb : 4l. 18 s. = 98 s. :: 8 lb : —

Then $98 \times 8 = 784.$ And 56) 784 (= 14 s. The Answer.

Note, The three last Questions are only the second varied, being proposed purely to give an Instance how any Question in this *Rule of Three* may be varied, according to page 86.

Quest. 6. What will three quarters of a Yard of Velvet cost, when the Price of 21 Yards and a half is worth 22l. 10 s. 6 d. This Question truly stated will stand

Thus, $21 \frac{1}{2} yds : 22l. 10 s. 6 d. :: \frac{3}{4}$ to the Answer.

Which may be found three several Ways; *viz.* by *Reduction*; by *Vulgar Fractions*; and by *Decimals*.

i. By *Reduction.* Bring the first and third Terms into one Denomination; *viz.* into Quarters, and reduce the second Term into it's least Denomination, *per Sect. 4. page 42.*

Thus $21 \frac{1}{2} = 86$ Quarters. And $22l. 10 s. 6 d. = 5406 Pence.$

Then $86 : 5406 :: 3 : 15 s. 8 \frac{1}{2} d.$ For $5406 \times 3 = 16218.$

And

And 86) 162 18 (= 188 $\frac{5}{8}$ d. Then 188 $\frac{5}{8}$ Pence = 15 s. 8 d. 2 $\frac{1}{4}$ Farthings; the Answer required.

2. The same Question stated in *Vulgar Fractions* will stand thus; $21 \frac{1}{2} = \frac{42}{2} : 22 \frac{3}{4} = \frac{90}{4} :: \frac{3}{4} :$ (See *Sett. 3. page 50.*)

Then $\frac{90}{4} \times \frac{3}{4} = \frac{270}{16}$. And $\frac{42}{2} \frac{270}{16} (= \frac{5670}{16}$ page 55, 56.

These $\frac{5670}{16}$ Parts of a Pound are brought into *Shillings* by multiplying the Numerator with 20, and dividing the Product by it's Denominator, &c.

Thus $5406 \times 20 = 108120$. And 6880) 108120 (15s.

And there remains 4920. Again, $4920 \times 12 = 59040$.

Then 6880) 59040 (8 d. and $\frac{5}{8}$, as before.

3. The same wrought by *Decimal Fractions* will be thus;

$21 \frac{1}{2} = 21,5$; $22 \text{ l. } 10 \text{ s. } 6 \text{ d.} = 22,525$, and $\frac{3}{4} = 0,75$

Therefore $21,5 : 22,525 :: 0,75 :$ to the Answer.

Then $22,525 \times 0,75 = 16,89375$

And $21,5$ 16,89375 (0,7857 l. = 15 s. 8 d. 2 far. $\frac{27}{1000}$.

Quest. 7. If 2 C. 3 qrs. 21 lb. of Sugar cost 6 l. 1 s. 8 d. What will 12 C. 2 qrs. cost at the same Rate?

That is, 2 C. 3 qrs. 21 lb. : 6 l. 1 s. 8 d. :: 12 C. 2 qrs. To what?

4	20	4
11 qrs.	121 s.	50 qrs.
28	12	28
88	250	1400 lb.
22	121	

Viz. $308 + 21 = 329 \text{ lb} : 1400 \text{ d.} :: 1400 \text{ lb} : \text{---}$

Then $1400 \times 1400 = 2044000$. And 329) 2044000 (6212 $\frac{1}{4}$ d. = 25 l. 17 s. 8 $\frac{1}{4}$ d. the Answer required.

The same Question stated in *Decimals* will stand thus;

$2,9375 : 6,0833 :: 12,5 :$ To the Answer.

Then $6,0833 \times 12,5 = 76,04125$ which being divided by 2,9375 will give 25,8863, &c. the Answer in *Decimals*, which brought into Coin, will be 25 l. 17 s. 8 $\frac{1}{4}$ d. as before.

Note, When the first Term is an Unit or 1, the Question is answered by Multiplication only.

Example. Suppose I give 5 *Shillings* 4 *Pence* for one Ounce of Silver, What must I pay for 32 $\frac{1}{2}$ Ounces at the same Rate?

That is 1 Ounce : 5 s. 4 d. :: 32 $\frac{1}{2}$ Ounces : To, &c. Which is best stated thus 1 : 64 d. :: 32,5 :

N

Then

Then $32,5 \times 64 = 2080 d. = 8l. 13s. 4d.$ the Answer required. For 1 neither multiplies nor divides.

When the second or third Term is an Unit or 1, then the Question is answered by Division only. As in this *Example*.

If a Silver Tankard weighing 21 Ounces, cost 5*l.* 19*s.* What is that an Ounce?

Thus $21 \text{ oz.} : 5 \text{ l. } 19 \text{ s.} = 119 \text{ s.} :: 1 : 5 \text{ s. } 8 \text{ d.}$ the Answer.
That is $21 \mid 119 (= 5 \text{ s. } \frac{14}{21} = 5 \text{ s. } 8 \text{ d.})$

The Proof of all Questions in the *Rule of Three Direct*, may be easily conceived from what hath been already said; *viz.* That the Product of the first and fourth Terms, must always be equal to the Product of the second and third Terms.

Or otherwise, by varying the Question, as in the second, third, fourth, and fifth Questions.

I shall conclude this Section with inserting a few Questions and their Answers; leaving their Work for the Learner's Practice.

Quest. 1. What will the Carriage of 17 *C.* 3 *qrs.* 11 *lb.* come to, at the Rate of 7*s.* the Hundred?

Answer 6*l.* 4*s.* 11 $\frac{1}{4}$ *d.*

Quest. 2. If 6*l.* 4*s.* 11 $\frac{1}{4}$ *d.* be paid for the Carriage of 17 *C.* 3 *qrs.* 11 *lb.*; What was paid for the Carriage of 1 *lb.*?

Answer 3 Farthings.

Quest. 3. A Grocer bought 3 *C.* 1 *qr.* 14 *lb.* Weight of Cloves, at the Rate of 2*s.* 4*d.* per Pound, and sold them for 52*l.* 14*s.* Whether did he gain or lose by the Bargain, and how much?

Answer, he gained 8*l.* 12*s.*

Quest. 4. A Draper bought of a Merchant eight Packs of Cloth; every Pack had four Parcels in it; and each Parcel contained ten Pieces; every Piece was Twenty-six Yards; he gave after the Rate of four Pounds sixteen Shillings for 6 Yards. What came the eight Packs to, and what were they worth per Yard?

Answ. They came to 6656*l.* And were worth 16*s.* per Yard.

Quest. 5. A Merchant bought 436 Yards of Broad Cloth for 8*s.* 6*d.* per Yard; and sold it again for 10*s.* 4*d.* per Yard. What did he gain by the 436 Yards?

Answ. he gained 39*l.* 19*s.* 4*d.*

Quest.

Quest. 6. A Goldsmith bought a Wedge of Gold, which weighed 14 lb. 3 oz. 8 pw. for 514 l. 4 s. What did he pay per Ounce?

Ans. 3 l. per Ounce.

Quest. 7. What will 48 oz. 17 pw. 20 Grains of Silver Plate come to, at the Rate of 5 s. 6 d. per Ounce?

Ans. 13 l. 1 s. 10 $\frac{3}{4}$ d.

Quest. 8. If in four Weeks one spend 13 s. 4 d. How long will 53 l. 6 s. last at that Rate?

Ans. 6 Years, 47 Days, 2 Hours, 24 $\frac{1}{2}$.

Quest. 9. What will the one eighth Part of a Ship be worth, when the half is valued at 1015 l. 10 s.

Ans. 253 l. 17 s. 6 d.

Quest. 10. The Sun is said to perform one entire Revolution, (or 360 Degrees) in the Space of 365 Days, 5 Hours, 48 Minutes, and 57 Seconds of Time, called a *Tropical* or *Solar Year*; How much doth it move in one Day?

Ans. 59 $\frac{1}{2}$. 8 $\frac{11}{16}$. 19 $\frac{111}{128}$ &c.

Quest. 11. If $\frac{5}{8}$ of a Yard of Velvet cost $\frac{2}{3}$ of a Pound Sterling, What will $\frac{1}{16}$ of a Yard cost of the same Velvet at that Rate?

Ans. $\frac{16}{240} = 1 s. 4 d.$

Quest. 12. Suppose 2 l. and $\frac{3}{8}$ of $\frac{1}{3}$ of a Pound Sterling will buy 3 Yards and $\frac{2}{3}$ of $\frac{3}{5}$ of a Yard of Cloth, How much will $\frac{3}{4}$ of a Yard cost at that Rate?

Ans. $\frac{2295}{480}$ of a Pound = 9 s. 4 $\frac{1}{2}$ d.

Sect. 2. Of Reciprocal Proportion; usually called The Rule of Three Inverse.

Reciprocal Proportion is, when of four Numbers the third (*viz.* that which moves the Question) beareth the same Ratio to the first: As the second does to the fourth.

Therefore, the less the third Term is, in respect to the first; the greater will the fourth Term be, in respect to the second.

EXAMPLE I.

If sixteen Men can do a Piece of Work in six Days; How many Days must eight Men require to do the same Work, at the same Rate of working?

Here it is plain that eight Men must needs have more Time than 16 Men to do the same Work. Consequently the greater

the third Term is, in respect to the first, the lesser will the fourth Term be, in respect to the second.

Example 2. If 8 Men can do a Piece of Work in 12 Days, How many Days will 16 Men require to do the same Work? Here it is plain the fourth Term must be less than the second, because 16 Men undoubtedly can do the same Work in less Time than 8 Men can.

From these Considerations, compared with those in page 85. it will be easy to perceive, whether the Terms of any proposed Question are in Direct or Reciprocal Proportion.

For when, according to the true Meaning and Design of any Question in Proportion, More requires More, or Less requires Less, the Terms are in Direct Proportion; as in this last Section.

But if More require Less, or Less require More (as above) then the Terms will be in Reciprocal Proportion.

The Manner of placing down the proposed Terms is the same in both Rules, viz. The first Term in the Supposition must be of the same Kind and Denomination with the third Term which moves the Question; and the Term sought must be of the same Kind and Denomination with the second Term in the Supposition. As in the two last Examples.

		<i>Men</i>	<i>Days</i>	<i>Men</i>	<i>Days</i>
Thus, is	{	<i>Example 1.</i>	16 : 6 ::	8 : —	
	{	<i>Example 2.</i>	8 : 12 ::	16 : —	

The Question being truly stated, observe this *Theorem*.

Theorem. { *Multiply the first and second Terms together, and divide the Product by the third Term, the Quotient will be the Answer required.*

Thus in the second *Example* $12 \times 8 = 96$.

Then 16) 96 (= 6 Days the Answer required.

That is, 16 Men may do the same Work in 6 Days, as 8 Men can do in 12 Days.

Now the Reason of this Operation (and consequently of the *Theorem*) is grounded upon this Consideration; viz. If 8 Men require 12 Days to do the Work, it is plain that one Man would require 8 Times 12 Days = 96 Days to do the same Work; but if one Man can do it in 96 Days, most certain 16 Men can do it in one 16th Part of that Time. Therefore 96 divided by 16 will give the Answer required, viz. 16) 96 (6 as before, &c.

Quest. 3. Suppose 800 Soldiers were besieged in a Town, and their Victuals were computed to serve them two Months (or 56 Days) How many of those Soldiers must depart the Garrison, that the same Victuals may serve the remaining Soldiers 5 Months.

The

The Question truly stated will stand

	<i>Months</i>		<i>Soldiers</i>		<i>Months</i>		<i>Soldiers</i>
Thus,	2	:	800	::	5	:	—
			2				

5) 1600 (320 : So many Soldiers may stay in the Garrison.

Consequently, $800 - 320 = 480$ Soldiers that must go out of the Garrison, which is the Answer required.

Question 4. *A* borrowed of his Friend *B* 250 *l.* for six Months, promising to do him the like Kindness upon Demand : Some Time after *B* desires *A* to lend him 400 *l.* the Question is, how long *B* must keep the 400 *l.* to be fully satisfied for his former Kindness to *A*.

Thus, 250 *l.* : 6 Months :: 400 *l.* : —

400) 1500 (3 Months.
12

3
28 Days in one Month.

4) 84 (21 Days. Answ. 3 Months, 21 Days.

Question 5. If a Penny White Loaf ought to weigh eight Ounces *Troy Weight*, when Wheat is sold for six Shillings Six-Pence the Bushel, what must it weigh when Wheat is sold for four Shillings the Bushel?

Thus 6 *s.* 6 *d.* = 78 *d.* : 8 *oz.* :: 4 *s.* = 48 *d.* : to the Answer.

48) 624 (13 *oz.* the Answer required.
48
144
144
(0)

The Proof of this Inverse Rule is easily deduced from it's Operations ; *viz.* The Product of the first and second Terms, must be equal to the Product of the third and fourth Terms.

Note, Any Question that falls under this Inverse Rule or Reciprocal Proportion, may be so stated as to have it's Terms in Direct Proportion ; by only changing the Places of the first and third Terms in the Question, Thus,

Question

Question 6. If a Field will feed eighteen Horses for seven Weeks: How long will it feed Forty-two Horses at the same Rate of feeding?

First, 18 Horses : 7 Weeks :: 42 Horses : 3 Weeks.

Here the Terms are stated inversely, as before.

Otherwise thus, 42 Horses : 7 Weeks :: 18 Horses : 3 Weeks. Then $18 \times 7 = 126$. And $126 \div 42 = 3$ Weeks. The Answer required.

Sect. 3. Of Compound Proportion; commonly called The Double Rule of Three.

Compound Proportion (as it is here meant) is, when there are five Numbers given to find out a sixth Proportional; and this is generally performed by a Double Position; that is, by stating and working the Question at two Operations, either in Direct or Reciprocal Proportion, according as the Question requires.

And therefore it is called, The Double Golden Rule, or Double Rule of Three.

The Double Rule Direct is, when the sixth Term or Number sought, is found by two Operations, both of them in Direct Proportion.

Example 1. If a Hundred Pounds gain six Pounds Interest in twelve Months; how much will three Hundred Pounds gain in nine Months, at the same Rate?

First 100*l.* : 6*l.* :: 300*l.* : 18*l.*

6
 $\frac{100 \mid 1800 \mid 18 \text{ l.}}{100} \left\{ \begin{array}{l} \text{The Interest of 300 l.} \\ \text{for twelve Months.} \end{array} \right.$

Months *Months*
 Then, 12 : 18*l.* :: 9 : 13*l.* 10*s.*

9
 $\frac{12 \mid 162 \mid 13 \text{ l. } 10 \text{ s.}}{12} \text{ The Answer required.}$

I suppose the Learner will easily conceive the Reason of these two Operations. For, first it is plain by Direct Proportion, that if 100*l.* gain 6*l.* in twelve Months, 300*l.* will gain 18*l.* in the same Time, and at the same Rate.

And

And by the same Rule it is plain, that if 12 Months will produce or give 18 *l.* Interest for 300 *l.* then 9 Months must needs give 13 $\frac{1}{2}$ for the same Sum, *viz.* 300 *l.*

The Double Rule of Three Inverse is, when the sixth Term, or Number sought, is found at two Operations (as before). But one of them requires an Answer in Reciprocal Proportion.

Question 2. If 6 Bushels of Oats will serve 4 Horfe 8 Days, How many Days will 21 Bushels serve 16 Horses, at the same Rate of feeding?

This *Question* being parted into two Positions, the first will be thus :

If 6 Bushels of Oats will serve 4 Horses 8 Days, How many Days will 21 Bushels serve them?

Here it is plain, that 21 Bushels will serve them longer than 6 Bushels ; therefore the first Position falls in Direct Proportion.

$$\begin{array}{r} \text{Bush. Days} \quad \text{Bush. Days} \\ \text{Thus, } 6 : 8 :: 21 : 28 \\ \quad \quad \quad 8 \\ \hline \quad \quad 6) 168 \text{ (28 Days)} \end{array}$$

That is, if 6 Bushels will serve 4 Horses 8 Days, 21 Bushels will serve them 28 Days.

The next Position must be to find how long the said 21 Bushels will serve 16 Horses at the same Rate of feeding : it is plain, that 21 Bushels cannot serve 16 Horses so many Days as they will serve 4 Horses ; therefore this second Position falls in Reciprocal Proportion.

$$\begin{array}{r} \text{Horses Days} \quad \text{Horses Days} \\ \text{Thus, } 4 : 28 :: 16 : 7 \text{ the Answer required.} \end{array}$$

After the like manner any Question in the Double Rule of Three may be answered by two single Positions, if Care be taken in stating them right, *viz.* Whether their Operation must be performed by the single Rule Direct, or Inverse.

But all Questions in this Double Rule, where five Numbers are proposed to find a sixth, may more easily and readily be answered by one general Theorem ; which compriseth both the Direct and Inverse Rules ; without considering either of them being deduced from the single Operations before-going.

But first you must carefully note, that in all Questions of this Nature, three of the five proposed Terms are always conditional and

and supposed; and that the other two move the Question. As for Instance in *Example 1.*

Viz. If 100*l.* will gain 6*l.* in 12 Months; these three Terms are only supposed or conditional. Then come the Question; What will 300*l.* gain in 9 Months? Now, in Order to raise the general Theorem, let us suppose, instead of Numbers, these Letters.

Viz. Let $\left\{ \begin{array}{ll} P = 100. & \text{The Principal.} \\ T = 12. & \text{The Time.} \\ G = 6. & \text{The Gain.} \end{array} \right\}$ In the Supposition of any proposed Question.

And, $\left\{ \begin{array}{ll} p = 300. & \text{The Principal.} \\ t = 9. & \text{The Time.} \\ g = 13,5. & \text{The Gain.} \end{array} \right\}$ The three Terms wherein the Question lies.

The $P : G :: p : \frac{Gp}{P} = \left\{ \begin{array}{l} \text{The Product of the two Means divided by the first Extreme.} \end{array} \right.$

That is, $100 : 6 :: 300 : \frac{300 \times 6}{100} = 18.$ $\left\{ \begin{array}{l} \text{Which is the first Part of the Question.} \end{array} \right.$

Then $T : \frac{Gp}{P} :: t : g$ $\left\{ \begin{array}{l} \text{Which is the second Part of the Question.} \end{array} \right.$

Viz. 12 : 18 :: 9 : 13,5

Ergo $Tg = \frac{Gpt}{P}$ $\left\{ \begin{array}{l} \text{That is, the Product of the Extremes is equal to that of the Means.} \end{array} \right.$

Consequently, $TgP = Gpt$ is the *Theorem*.

This *Theorem* affords two Rules, by which all Questions in this Double Rule of Three, or rather of five Numbers, may be resolved; due Regard being had to the true placing down of the proposed Terms, which must be thus:

Always place the three conditional Terms in this Order; let that Number which is the principal Cause of Gain, Loss, or Action, &c. (*viz.* *P.*) be put in the first Place; that Number which denotes the Space of Time, or Distance of Place, &c. (*viz.* *T.*) be put in the second Place. And that Number which is the Gain, Loss, or Action, &c. (*viz.* *G.*) be put in the third Place. Now according to these Directions, the conditional Terms of the last Question will stand thus; *P. T. G.*

That done, place the other two Terms which move the Question, underneath those of the same Name,

Thus, $\left\{ \begin{array}{l} P. T. G. \\ p. t. \end{array} \right.$

Then

Then if the Blank or Term sought, fall under the third Place as in this Question,

It will be $\left\{ \frac{G p t}{T P} = g. \right.$ Which gives this Rule.

Rule 1. $\left\{ \begin{array}{l} \text{Multiply the three last Terms together for a Dividend,} \\ \text{and the two first together for a Divisor; the Quotient} \\ \text{arising from them will be the sixth Term.} \end{array} \right.$

That is, in our proposed *Example 1.*

Thus $6 \times 300 \times 9 = 16200$ the Dividend.

And $100 \times 12 = 1200$ the Divisor.

Then $1200) 16200 (13\frac{1}{2}$ the Answer, as before.

But if the Blank or Term sought fall under the first Place, then

It will be $\left\{ \frac{T g P}{t G} = p. \right.$

Or if the Blank fall under the second Place,

It will be $\left\{ \frac{T g P}{G p} = t. \right.$ Either of these give this Rule.

Rule 2. $\left\{ \begin{array}{l} \text{Multiply the first, second, and last Terms together for} \\ \text{a Dividend, and the other two together for a Divisor;} \\ \text{the Quotient arising from them will be the sixth Term.} \end{array} \right.$

And because our *Example 2.* falls under the Consideration both of Direct and Reciprocal Proportion, let it be here proposed again.

Viz. If 6 Bushels of Oats will serve 4 Horses 8 Days; how many Days will 21 Bushels serve 16 Horses, &c.

If the Terms of this Question be placed down as before directed, they will stand

	Horses.	Days.	Bushels.	
Thus	$\left\{ \begin{array}{c} 4 \\ 16 \end{array} \right.$	$\cdot 8 \cdot$	$\begin{array}{c} 6 \\ 21 \end{array}$	Terms in the Supposition.

Here the Blank falls under the second Place, therefore it must be found by the second Rule.

Thus $4 \times 8 \times 21 = 672$ the Dividend.

And $16 \times 6 = 96$ the Divisor.

Then $96) 672 (7$ the Answer, as before.

Quest. 3. What Principal or Stock will gain 20 *l.* in 8 Months at 6 *per Cent. per Annum*?

<i>Prin.</i>	<i>Time.</i>	<i>Gain.</i>
100	12	6
	8	20

Terms in the Supposition.

In this Question the Blank falls under the first Place, therefore it must be found by the second Rule.

Thus $100 \times 12 \times 20 = 24000$ the Dividend.

And $8 \times 6 = 48$ the Divisor.

Then $48 \overline{) 24000}$ (500 *l.* the Answ. required.

The Proof of all Questions in this Double Rule of five Numbers, is best performed by varying the Question; *viz.* by stating it in another Order, as in the last *Example*: Thus,

If 100 *l.* gain 6 *l.* in 12 Months, what will 500 *l.* gain in 8 Months?

The Answer to this Question must be 20 *l.* if the Work of the last *Example* be true.

	<i>Prin.</i>	<i>Time.</i>	<i>Gain.</i>
Stated thus	100	12	6
	500	8	

} then, *per Rule 1,*

$500 \times 8 \times 6 = 24000$. And $100 \times 12 = 1200$.

Then $1200 \overline{) 24000}$ (20 *l.* the Answer, &c.

Quest. 4. If two Men can do 12 Rods of Ditching in 6 Days, How many Rods may be done by 8 Men in 24 Days, at the same Rate of working?

Answ. 192 Rods.

Quest. 5. If the Carriage of 5 C. 3 qrs. Weight, 150 Miles, cost 3 *l.* 7 *s.* 4 *d.* What must be paid for the Carriage of 7 C. 2 qrs. 25 lb. Weight, 64 Miles, at the same Rate?

Answ. 1 *l.* 18 *s.* 7 $\frac{1}{2}$ *d.*

Quest. 6. If 8 Men deserve 2 *l.* Wages for 5 Days Work, How much will 32 Men deserve for 24 Days, at the same Rate?

Answ. 38 *l.* 8 *s.*

Quest. 7. Suppose a Hundred Pounds would defray the Expences of five Men for Twenty-two Weeks and six Days, How long would twelve Men be in spending of one Hundred and Fifty Pounds, at the same Rate?

Answ. 14 Weeks and 2 Days.

C H A P.

C H A P. VIII.

Of Trading in Company, usually called the Rule of Fellowship; also Bartering, and Exchanging of Coins, &c.

THE *Rule of Fellowship* is that by which the Accompts of several Partners trading in a Company, are so adjusted or made up, that every Partner may have his just Part of the Gain, or sustain his just Part of the Loss; according to the Proportion or Share of Money he hath in the Joint-Stock: Now this falls under two Considerations, called the *Single* and *Double Rules of Fellowship*.

Se&t. 1. *The Single Rule of Fellowship, viz. That without Time.*

BY the *Single Rule of Fellowship* is adjusted the Accompts of those Partners that put all their several and perhaps different Sums of Money, into a common Stock at one and the same Time; and therefore it is usually called the *Rule of Fellowship* without Time: Now all Questions of this Nature are answered by so many several Operations in the *Rule of Three Direct*, as there are Partners in the Stock.

For, as the Total Sum of Money in the Stock is in Proportion to the whole Gain, or Loss: so is every Man's particular Part of that Stock; to his particular Share of that Gain, or Loss.

Quest. 1. Three Partners, suppose *A*, *B*, and *C*, make a Joint-Stock of 96 *l.* in this manner.

A, puts in 24 *l.* *B*, puts in 32 *l.* and *C*, puts in 40 *l.* with this 96 *l.* they trade and gain 12 *l.* It is required to find each Man's true Part of that Gain.

The Operation will stand, thus

$$96\text{ l.} : 12\text{ l.} :: \left\{ \begin{array}{l} 24\text{ l.} : 3\text{ l.} = A's \\ 32\text{ l.} : 4\text{ l.} = B's \\ 40\text{ l.} : 5\text{ l.} = C's \end{array} \right\} \text{ Part of the Gain.}$$

Proof $3\text{ l.} + 4\text{ l.} + 5\text{ l.} = 12\text{ l.}$ the whole Gain.

That is, if the Sum of each Man's particular Gain, amount to the whole Gain, the Work is true; if not, some Error is committed which must be found out.

Note, These Operations will be very much abbreviated, if you work them by *Theorem 2.* page 87. For here 96 is a common Antecedent, and 12 is the common Consequent in all the three Proportions.

Therefore $96 : 12 :: 1 : 0,125$ a common Multiplier.

$$\text{Then } \left\{ \begin{array}{l} 24 \\ 32 \\ 40 \end{array} \right\} \times 0,125 = \left\{ \begin{array}{l} 3\text{ l.} \\ 4\text{ l.} \\ 5\text{ l.} \end{array} \right\} \text{ for } \left\{ \begin{array}{l} A, \\ B, \\ C, \end{array} \right\} \text{ as before.}$$

Now this Method is more readily performed than the other, especially when the Partners are many ; because one Single Division serves for all the Work.

Quest. 2. Three Merchants, *A*, *B*, and *C*, freight a Ship with 248 Tons of Wine : Thus, *A*, loaded 98 Ton, *B*, 86 Ton, and *C*, 64 Ton. By Extremity of Weather the Seamen were forced to cast or throw 93 Ton of it over-board. How much of this Loss must each Merchant sustain ?

First, $248 : 93 :: 1 : 0,375$ the common Multiplier.

$$\text{Then } \left\{ \begin{array}{l} 98 \\ 86 \\ 64 \end{array} \right\} \times 0,375 = \left\{ \begin{array}{l} 36,75 \text{ for } A's \\ 32,25 \text{ for } B's \\ 24,00 \text{ for } C's \end{array} \right\} \text{ Loss.}$$

Proof $93,00 =$ the whole Loss.

Now if the Question were to find how much of the remaining Wine that was saved, belongs to *A*, to *B*, and to *C*.

$$\text{Then } \left\{ \begin{array}{l} 98 - 36,75 = 61,25 \\ 86 - 32,25 = 53,75 \\ 64 - 24,00 = 40,00 \end{array} \right\} \text{ belongs to } \left\{ \begin{array}{l} A. \\ B. \\ C. \end{array} \right.$$

That is, *A* ought to have 61 Tons and 63 Gallons. *B*, ought to have 53 Tons and 189 Gallons. And *C*, ought to have 40 Tons of what was left.

Quest. 3. Suppose six Men, viz. *A*, *B*, *C*, *D*, *E*, and *F*, make a Joint-Stock of 2558 l.

	l.	s.	Decimals.
Thus $\left\{ \begin{array}{l} A \\ B \\ C \\ D \\ E \\ F \end{array} \right\}$ puts in	654	. 10	= 654,50
	543	. 15	= 543,75
	480	. 00	= 480,00
	254	. 10	= 254,50
	365	. 05	= 365,25
	260	. 00	= 260,00

The whole Stock - 2558 . 00 = 2558,00 according to the Question.

With

With this Stock of 2558*l.* they Trade eighteen Months, and Gain 831*l.* 7*s.* It is required to find every Man's Part or Share of that Gain.

Note, *Although the Time of Trading, viz. eighteen Months, be mentioned in the Question, yet it is no Way concerned in answering of it; as you may observe in the following Work.*

First, 2558*l.* : 831,35*l.* :: 1*l.* : 0,325 Decimal Parts.
Consequently, 1*l.* : 0,325 :: 654,5 : 212,7125. That is,

$$\left. \begin{array}{r} 654,50 \\ 543,75 \\ 480,00 \\ 254,50 \\ 365,25 \\ 260,00 \end{array} \right\} \times 0,325 = \left\{ \begin{array}{r} 212,71250 \\ 176,71875 \\ 156,00000 \\ 82,71250 \\ 118,70625 \\ 84,50000 \end{array} \right\} \text{ for } \left\{ \begin{array}{l} A. \\ B. \\ C. \\ D. \\ E. \\ F. \end{array} \right.$$

	<i>l.</i>	<i>Parts.</i>	<i>l.</i>	<i>s.</i>	<i>d.</i>
That is, $\left\{ \begin{array}{l} A. \\ B. \\ C. \\ D. \\ E. \\ F. \end{array} \right\}$ gains	$\left\{ \begin{array}{l} 212,71250 \\ 176,71875 \\ 156,00000 \\ 82,71250 \\ 118,70625 \\ 84,50000 \end{array} \right\}$	$=$	$\left\{ \begin{array}{l} 212 \\ 176 \\ 156 \\ 82 \\ 118 \\ 84 \end{array} \right\}$	$\left\{ \begin{array}{l} 14 \\ 14 \\ 00 \\ 14 \\ 14 \\ 10 \end{array} \right\}$	$\left\{ \begin{array}{l} 03 \\ 04\frac{1}{2} \\ 00 \\ 03 \\ 01\frac{1}{2} \\ 00 \end{array} \right\}$
Proof. Sum	831,35	$=$	831	07	00

I have omitted resolving this Question according to the usual Method (as before directed) of finding every Man's particular Part of the Gain by the Golden Rule, as in the first Work of *Example 1.* leaving that for the Learner's Practice.

Sect. 2. *The Double Rule of Fellowship; or that with Time.*

THIS is usually called the *Double Rule of Fellowship*, because every particular Man's Money is to be considered with Relation to the Time of it's Continuance in the Joint-Stock.

Question 1. *A*, and *B*, join in Partnership upon these Terms, viz. *A*, agrees to lay down 100*l.* and to employ it in Trade 3 Months: Then *B*, is to lay down his 100*l.* and with the whole Stock of 200*l.* they are to trade 3 Months more. Now at the End of that Time, they find their whole Gain to be 21*l.* It is required to know what each Man's Part of the Gain ought to be, according to his Stock, and the Time of employing it.

Here

Here it is but reasonable to conclude, that *A*, ought to gain more than *B*, notwithstanding their Stocks of Money are equal; because *A* employed his Money a longer Time than *B*.

Now for solving of this Question, let us suppose *A*'s 100 *l.* employed the first 3 Months to gain $Z =$ a Sum as yet unknown; then it must gain $2Z$ in 6 Months; and to find what *B*, must gain, it will be,

$$\begin{array}{lcl} \text{1.} & \text{Months.} & \\ 100 & . \ 6 & . \ 2Z = A's \text{ Gain} \\ 100 & . \ 3 & . \quad \quad \text{to } B's \text{ Gain} \end{array} \left. \vphantom{\begin{array}{l} 100 \\ 100 \end{array}} \right\} \text{per Rule 1. Page 97.}$$

$$\text{Ergo } \frac{100 \times 3 \times 2Z}{100 \times 6} = B's \text{ Gain.}$$

But *A*'s Gain added to *B*'s Gain must $= 21l.$ the whole Gain by the Question.

$$\text{Therefore } 2Z + \frac{100 \times 3 \times 2Z}{100 \times 6} = 21l.$$

That is, $100 \times 6 \times 2Z + 100 \times 3 \times 2Z = 21 \times 100 \times 6$.
Which contracted is, $900 \times 2Z = 21 \times 600$.

Consequently, $2Z = \frac{21 \times 600}{900}$, which gives the following Analogy.

Viz. $900 : 21 :: 600 : 2Z = 14l.$ for *A*'s Gain.

And $900 : 21 :: 100 \times 3 = 300 : 7l.$ for *B*'s Gain.

Now this way of arguing hath not only resolved the present Question, but it also affords (and demonstrates) a general Rule for resolving all Questions of this Nature, be the Partners never so many.

Rule. $\left\{ \begin{array}{l} \text{Multiply every particular Man's Stock, with the Time it} \\ \text{is employed, then it will be, as the Sum of all those} \\ \text{Products; is to the whole Gain (or Loss). So is every} \\ \text{one of those Products: to it's proportional Part of that} \\ \text{whole Gain (or Loss).} \end{array} \right.$

Question 2. Three Merchants *A*, *B*, and *C*, enter into Partnership, thus; *A* puts into the Stock 65 *l.* for 8 Months; *B* puts in 78 *l.* for 12 Months; and *C* puts in 84 *l.* for 6 Months. With these they traffick, and gain 166 *l.* 12 *s.* It is required to find each Man's Share of the Gain, proportionable to the Stock and Time of employing it.

1. *A*'s

$$\begin{array}{l}
 1. A's \\
 2. B's \\
 3. C's
 \end{array}
 \left. \vphantom{\begin{array}{l} 1. A's \\ 2. B's \\ 3. C's \end{array}} \right\} \text{Stock}
 \left\{ \begin{array}{l} 65 \text{ l.} \times 8 \\ 78 \text{ l.} \times 12 \\ 84 \text{ l.} \times 6 \end{array} \right\}
 \begin{array}{l}
 \text{Months, the Time it was} \\
 \text{employed} =
 \end{array}
 \left\{ \begin{array}{l} 520 \\ 936 \\ 504 \end{array} \right.$$

The Sum of those Products is, 1960

Then, according to the Rule, the several Proportions will stand thus,

$$1960 : 166,6 :: \left\{ \begin{array}{l} 520 : 44,20 = 44 \text{ l. } 4 \text{ s. } 0 \text{ d.} \\ 936 : 79,56 = 79 \text{ l. } 11 \text{ s. } 2 \frac{1}{2} \text{ d.} \\ 504 : 42,84 = 42 \text{ l. } 16 \text{ s. } 9 \frac{1}{2} \text{ d.} \end{array} \right\} \text{ for } \left\{ \begin{array}{l} A. \\ B. \\ C. \end{array} \right.$$

The whole Gain = 166 l. 12 s. 0 d.

Or you may work as in some of the former *Examples*, viz. by finding the proportional Part of the Gain due to one Pound, &c.

Thus 1960 : 166,6 :: 1 : 0,085 the common Multiplier.

$$\text{Then } \left\{ \begin{array}{l} 520 \\ 936 \\ 504 \end{array} \right\} \times 0,085 = \left\{ \begin{array}{l} 44,2 \\ 79,56 \\ 42,84 \end{array} \right\} \text{ for } \left\{ \begin{array}{l} A. \\ B. \\ C. \end{array} \right\} \text{ \&c. As before.}$$

Question 3. Six Merchants, viz. *A, B, C, D, E, and F*, enter into Partnership, and compose a Joint-Stock in this manner;

$$\text{Viz. } \left\{ \begin{array}{l} A \\ B \\ C \\ D \\ E \\ F \end{array} \right\} \text{ puts in } \left\{ \begin{array}{l} 64 \text{ l. } 10 \text{ s.} \\ 78 \text{ l. } 15 \text{ s.} \\ 100 \text{ l. } 00 \text{ s.} \\ 80 \text{ l. } 10 \text{ s.} \\ 74 \text{ l. } 12 \text{ s.} \\ 125 \text{ l. } 15 \text{ s.} \end{array} \right\} \text{ for } \left\{ \begin{array}{l} 4 \frac{1}{2} \\ 6 \\ 8 \frac{1}{4} \\ 12 \\ 9 \frac{1}{2} \\ 7 \end{array} \right\} \text{ Months.}$$

They traffick, and gain 258 l. 18 s. 4 $\frac{1}{2}$ d. It is required to find every Man's Share of the Gain, according to the Stock and Time it was employed.

The several Stocks of Money, and their respective Times being first brought into Decimals, and then multiplied together, will produce these following Products.

$$\begin{array}{l}
 A's \\
 B's \\
 C's \\
 D's \\
 E's \\
 F's
 \end{array}
 \left. \vphantom{\begin{array}{l} A's \\ B's \\ C's \\ D's \\ E's \\ F's \end{array}} \right\} \text{Stock}
 \left\{ \begin{array}{l} 64,50 \times 4,50 \\ 78,75 \times 6,00 \\ 100,00 \times 8,25 \\ 80,50 \times 12,00 \\ 74,6 \times 9,50 \\ 125,15 \times 7,00 \end{array} \right\}
 \begin{array}{l}
 \text{The Time it was} \\
 \text{employed} =
 \end{array}
 \left\{ \begin{array}{l} 290,25 \\ 472,50 \\ 825,00 \\ 966,00 \\ 708,70 \\ 880,25 \end{array} \right.$$

The Sum of those Products = 4142,70

Then

Then if you work by the common Way; it will be $4142,7 : 258,91875 :: 290,25 : 18,140625 = 18l. 2s. 9\frac{3}{4}d.$ for *A*'s part of the Gain; and so on for the rest.

But if you work by the easiest Way, *viz.* by finding the proportional Part of the Gain due to one Pound.

Thus $4142,7 : 258,91875 :: 1 : 0,0625.$

Then		<i>l.</i>	<i>s.</i>	<i>d.</i>	
290,25	} $\times 0,0625 =$	18,140625	=	18.02.09	$\frac{3}{4}$
472,50		29,531250	=	29.10.07	$\frac{1}{2}$
825,00		51,562500	=	51.11.03	
966,00		60,375000	=	60.07.06	
708,70		44,293750	=	44.05.10	$\frac{1}{2}$
880,25		55,015625	=	55.00.03	$\frac{3}{4}$
					} for $\begin{cases} A \\ B \\ C \\ D \\ E \\ F \end{cases}$

The whole Gain = $258.18.04\frac{1}{2}$

These few *Examples* being well understood, are sufficient to shew the whole Business of Fellowship, &c.

Sect. 3. Of Bartering.

WHEN Merchants, or Tradesmen, exchange one Commodity for another, it is called *Bartering*; and the only Difficulty in this way of dealing, lies in duly proportioning the Commodities to be exchanged, so as that neither Party may sustain Loss.

Question 1. Two Merchants, *A*, and *B*, *Barter*; *A* would exchange 5 *C.* 3 *qrs.* 14 pound of *Pepper*, which is worth 3*l.* 10*s.* per *C.* with *B* for *Cotton*, worth 10 *d.* per pound weight; how much *Cotton* must *B* give to *A* for his *Pepper*?

Note, In order to the resolving of this *Question* (and all other *Questions* of this Nature) you must first find, by the Rule of Three (or otherwise) the true Value of that Commodity whose Quantity is given (which in this *Question* is *Pepper*). And then find how much of the other Commodity will amount to that Sum, at the Rate proposed.

First 5 *C.* 3 *qrs.* 14 lb. = 5,875 } in Decimals.
And 3*l.* 10 *s.* 0 *d.* = 3,500 }

Then $1 : 3,5 :: 5,875 : 20,5625 = 20l. 11s. 3d.$ the true Value of the *Pepper*.

Next, it is easy to conceive, that *A* ought to have as much *Cotton* at 10 *d.* per Pound, as will amount to 20*l.* 11*s.* 3*d.* which may be thus found;

$10d. : 1lb. :: 20l. 11s. 3d. = 4235d. : 493,5lb.$

That

That is, 4 C. 1 qr, 17 $\frac{1}{2}$ pound of Cotton. And so much B must give to A in exchange for his 5 C. 3 qrs. 14 pound of Pepper.

Question 2. Two Merchants A and B barter thus; A hath 86 Yards of Broad Cloth worth 9s. 2d. per Yard ready Money: but in Barter he will have 11s. per Yard. B hath Shalloon worth 2s. 1d. per Yard ready Money; it is required to find how many Yards of the Shalloon B must give to A for his Cloth, to make his Gain in the Barter equal to that of A's.

The Method of resolving this, and the like Questions, differs a little from the last Case; for in this you must first find what Advance B ought to make per Yard upon his Shalloon, in proportion to what A hath done upon a Yard of his Cloth.

Thus $\left\{ \begin{array}{cccccccccccc} s. & d. & d. & s. & d. & s. & d. & d. & s. & d. & d. \\ 9. & 2 & = & 110 : 11 = & 132 :: & 2. & 1 = & 25 : 2. & 6 = & 30 \end{array} \right.$
the advanced Price for a Yard of B's Shalloon. Then proceed as before in the last Example.

Thus 1 Yard : 11s. :: 86 Yards : 946s. = 47l. 6s. the advanced Value of all the Cloth.

Next, If 2s. 6d. will buy one Yard of Shalloon, at it's advanced Price, how many Yards will 47l. 6s. buy.

Thus 2,5 : 1 :: 946 : 378,4 Yards.

That is, B must give 378 $\frac{2}{3}$ Yards of his Shalloon to A, for his 86 Yards of Broad Cloth.

These two Examples are sufficient to shew the Learner, that the Method of bartering, or exchanging Commodities for Commodities, wholly depends upon a clear understanding of the Golden Rule; which indeed is so called, because of it's Universal Use.

Sect. 4. Of Exchanging Coins.

EXchanging the Coins of one Country for those of another, is like the Business of bartering Commodities. That is, it consists in finding what Sum of one Country Coin will be equal in Value to any proposed Sum of another Country Coin. And, in order to perform that, it will be very necessary to have a true Account at all times of the just Value of those Foreign Coins which are to be exchanged, as they are compared in Value with our English Coin.

I say, at all times, because the Par of Exchange (as the Merchants call it) differs almost every Day from London to other Countries. That is, it rises and falls, according as Money is plenty or scarce; or according to the Time allowed for Payment of the Money in Exchange, &c.

P

Those

Those that desire to be fully satisfied in the common Values of Foreign Coins, Weights, Measures, &c. may find them in a Book called the *Merchants Map of Commerce*, which for Brevity sake I have omitted transcribing, and only collected these few of Coins.

Foreign Coins.		English Coin.		
		l.	s.	d.
French Coin.	A Denier =	0	0	0 $\frac{3}{40}$
	12 Deniers = 1 Soultz =	0	0	0 $\frac{2}{10}$
	12 Soultz = 1 Livre =	0	1	6
	3 Livres = 1 Crown =	0	4	6
Low-Country Coin.	A Stiver =	0	0	1 $\frac{1}{5}$
	6 Stivers = 1 Flemish Shilling =	0	0	7 $\frac{2}{5}$
	20 Stivers = 1 Gilder =	0	2	0
	10 Gilders = 33 $\frac{1}{3}$ Shillings } =	1	0	0
	or a Flemish Pound }			
	Embden Dollar =	0	2	3 $\frac{3}{5}$
	Campan Dollar =	0	2	7 $\frac{1}{5}$
	A { Zealand Dollar =	0	3	0
	Lyons Dollar =	0	4	0
	Specie Dollar =	0	5	0
Germany.	Duccatoon =	0	6	3 $\frac{2}{5}$
	A Rixdollar of the Empire =	0	4	5 $\frac{1}{4}$
	A Gilder of Nuremberg =	0	7	1
	The Livre at Leghorn =	0	0	9
	Florence Crown Current =	0	5	3
	Venice Ducat de Banco =	0	4	4
	The Current Ducat =	0	3	4
	The Naples Ducat =	0	5	0
	The Cadiz Ducat =	0	5	6 $\frac{1}{4}$
	The Barcelona Ducat =	0	6	0
In Italy and Spain.	The Valencia Ducat =	0	5	3
	The Bergonia Ducat =	0	4	4
	The Portugal Testoon =	0	1	3
	The Piece of Eight =	0	4	6

Note, The *English* generally reckon their Exchange with other Countries by Pence, viz. other Countries value their Crowns, Dollars, or Ducats, &c. by *English* Pence. Except with some Parts of the *Low-Countries*, with whom the Exchange is in Pounds Sterling.

Quest. 1. How many Dollars at 4s. 6d. per Dollar, may one have for 162l. 18s.

Answer 724 Dollars.

Thus

Thus $162\text{ l. } 18\text{ s.} = 3258\text{ s.}$ and $4\text{ s. } 6\text{ d.} = 54\text{ d.}$

Then $54 : 1 :: 3258 : 724$ the Answer.

Quest. 2. How many *Saragossa* Ducats, of $5\text{ s. } 6\text{ d.}$ the Ducat, may be had for 275 Bergonia Ducats, at $4\text{ s. } 4\text{ d.}$ the Piece?

Answer 216 and $3\text{ s. } 8\text{ d.}$ over.

Thus $5\text{ s. } 6\text{ d.} = 66\text{ d.}$ and $4\text{ s. } 4\text{ d.} = 52\text{ d.}$

Then $275 \times 52 = 14300\text{ d.} = 275\text{ Ducats.}$

Consequently $66) 14300$ ($216\frac{2}{3}$ the Answer required.

Quest. 3. A Traveller would change $233\text{ l. } 16\text{ s. } 8\text{ d.}$ Sterling Money; for *Venice* Ducats at $4\text{ s. } 9\frac{1}{2}\text{ d. per Ducat}$; How many Ducats must he have?

Answer 976 Ducats.

Thus $4\text{ s. } 9\frac{1}{2}\text{ d.} = 57,5\text{ d.}$ and $233\text{ l. } 16\text{ s. } 8\text{ d.} = 56120\text{ d.}$

Then $57,5\text{ d.}) 56120\text{ d.}$ (976 the Answer required.

Quest. 4. A Cashier hath received 759 Ducats, at $7\text{ s. } 6\text{ d. per Ducat}$; And 579 Dollars at $4\text{ s. } 8\text{ d. per Dollar}$: Which he would exchange for *Flemish* Marks at $14\text{ s. } 3\text{ d. per Piece}$: How many ought he to have?

Answer 589 Marks, and 15 d. over.

For $7\text{ s. } 6\text{ d.} = 90\text{ d.}$ and $4\text{ s. } 8\text{ d.} = 56\text{ d.}$

Then $\begin{cases} 759 \times 90 = 68310\text{ d.} \text{ the Value of the Ducats.} \\ 579 \times 56 = 32424\text{ d.} \text{ the Value of the Dollars.} \end{cases}$

their Sum $= 100734\text{ d.}$

And $14\text{ s. } 3\text{ d.} = 171\text{ d.}$ the *Flemish* Mark in Pence.

Consequently $171) 100734$ ($589\text{ \&c.}$ the Answer required.

Quest. 5. A Bill of Exchange was accepted at *London* for the Payment of 400 l. Sterling, for the like Value delivered in *Amsterdam*, at $1\text{ l. } 13\text{ s. } 6\text{ d.}$ for 1 l. Sterling; How much money was delivered at *Amsterdam*?

Answer. 670 l. Flemish.

For $1\text{ l.} = 240\text{ d.}$ and $1\text{ l. } 13\text{ s. } 6\text{ d.} = 402\text{ d.}$

Then $240 : 402 :: 400 : 670$ the Answer required.

Quest. 6. When the Exchange from *Antwerp* to *London* is at $1\text{ l. } 4\text{ s. } 7\text{ d. Flemish}$, for 1 l. Sterling; How many Pounds Sterling must be paid at *London*; to ballance 236 l. Flemish at *Antwerp*.

Answer 192 l. Sterling.

Thus $1\text{ l. } 4\text{ s. } 7\text{ d.} = 295\text{ d.}$ and $1\text{ l.} = 240\text{ d.}$

Then $295 : 240 :: 236 : 192$ the Answer.

Quest. 7. A Merchant delivered at *London* 120 *l.* Sterling to receive 147 *l.* *Flemish* at *Amsterdam*; How much was 1 *l.* Sterling valued at, in *Flemish* Money?

Answer. 1 *l.* 4 *s.* 6 *d.*

Thus $120 : 147 :: 240 \text{ d.} : 294 \text{ d.} = 1 \text{ l. } 4 \text{ s. } 6 \text{ d. } \&c.$

Quest. 8. A Factor hath sold Goods at *Cadiz* for 1468 Pieces of Eight, valued at 4 *s.* 6 $\frac{1}{2}$ *d.* Sterling per Piece; How much Sterling Money do those Pieces of Eight amount to?

Answer 333 *l.* 7 *s.* 2 *d.*

Thus, if 1 = 54,5 *d.* then $1468 \times 54,5 = 83006 \text{ d. } \&c.$

Quest. 9. A Traveller would have an equal Number of Crowns at 5 *s.* 6 *d.* per Crown; and Dollars at 4 *s.* 5 *d.* per Piece; How many of each sort may he have for 309 *l.* 8 *s.*?

Answer 624 of each.

Thus $309 \text{ l. } 8 \text{ s.} = 74256 \text{ d.}$

And $5 \text{ s. } 6 \text{ d.} + 4 \text{ s. } 5 \text{ d.} = 119 \text{ d.}$

Then $119 \overline{) 74256}$ (624 the Answer required.

Quest. 10. Suppose I would exchange 527 *l.* 17 *s.* 6 *d.* for Dollars at 4 *s.* 6 *d.* a Piece, Ducats at 5 *s.* 8 *d.* a Piece, and Crowns at 6 *s.* 1 *d.* a Piece; and would have 2 Dollars for 1 Ducat, and 3 Dollars for 2 Crowns. How many of each sort must I have?

Answer 927 Dollars, $463 \frac{1}{2}$ Ducats, and 618 Crowns.

For $\left\{ \begin{array}{l} 54 \text{ d.} = 1 \text{ Dollar.} \\ 68 \text{ d.} = 1 \text{ Ducat.} \\ 73 \text{ d.} = 1 \text{ Crown.} \end{array} \right\} \text{ per Question.}$

And $126690 \text{ d.} = 527 \text{ l. } 17 \text{ s. } 6 \text{ d.}$

Now if the Crowns, Dollars, and Ducats, were to be equal in Number; then $73 + 54 + 68$ must have been the Divisor, by which 126690 must have been divided, and the Quotient would have been the Answer to the Question. As in the last *Example*.

But here instead of their Sum, such Parts of them must be taken as are assigned or limited by the Question; that so the Number of some one of them may be found.

And because there must be $\left\{ \begin{array}{l} 2 \text{ Dollars for 1 Ducat, and} \\ 3 \text{ Dollars for 2 Crowns,} \end{array} \right.$

Therefore it will be $\frac{1}{2}$ of a Ducat for one Dollar, and $\frac{2}{3}$ of a Crown for one Dollar.

Consequently,

Consequently, $54 + \frac{58}{2} : + \frac{2}{3}$ of 73 = $136 \frac{2}{3}$, or $\frac{410}{3}$ will be the Divisor to find the Number of Dollars.

Thus $\frac{410}{3}$ 126690 (927 the Number of Dollars.

Then $\frac{1}{2}$ of 927 = $463 \frac{1}{2}$ is the Number of Ducats.

And $\frac{2}{3}$ of 927 = 618 is the Number of Crowns.

Or if you please you may form Divisors to find either the Ducats or Crowns first: For if it be 2 Dollars for 1 Ducat, and 3 Dollars for 2 Crowns, as before;

Then will 6 Dollars be for 3 Ducats, and 6 Dollars for 4 Crowns.

Therefore, $\left\{ \begin{array}{l} \frac{2}{3} \text{ of a Dollar} \\ \frac{3}{4} \text{ of a Ducat} \end{array} \right\}$ will be for 1 Crown.

Consequently, $\frac{2}{3}$ of 54 : $+ \frac{3}{4}$ of 68 : $+ 73 = 205$ will be the Divisor to find the Crowns first, &c.

Quest. 11. A Cashier is to receive 500*l.* He is offered Crowns at 6*s.* $1 \frac{1}{2}$ *d.* per Crown, which are worth but 6*s.* Or he may have Dollars at 4*s.* 5*d.* the Piece, which are worth but 4*s.* 4*d.* Which of these shall he receive to have the least Loss? And how much will he lose in the Payment?

1 $\left\{ \begin{array}{l} 1 \text{ Crown} = 72 \text{ d.} \\ 1 \text{ Dollar} = 52 \text{ d.} \end{array} \right\}$ according to their true Values.

2 $\left\{ \begin{array}{l} 1 \text{ Crown} = 73,5 \text{ d.} \\ 1 \text{ Dollar} = 53,0 \text{ d.} \end{array} \right\}$ the advanced Values.

Now to find which will be the least Loss; find what the advanced Value of a Dollar ought to be in Proportion to that of 1 Crown.

Thus $72 : 73,5 :: 52 : 53,083 \text{ \&c.}$ But he may have Dollars at 53*d.* per Piece, therefore the Payment in Dollars will be the least Loss; viz. 53 is less than 53,083 &c.

Next, to find what the whole Loss will be by receiving Dollars. Because the 500*l.* = 120000*d.* is advanced as much above the true Value, as 53*d.* is above 52*d.* Therefore say, If 53*d.* advance 1*d.* = 53*d.* — 52*d.*; what will 120000*d.* advance? *i. e.*

$53 \text{ d.} : 1 \text{ d.} :: 120000 \text{ d.} : 2264 \frac{8}{33} \text{ d.} = 9 \text{ l. } 8 \text{ s. } 4 \frac{8}{33} \text{ d.} =$ the Loss.

Quest. 12. Suppose I exchange 4*l.* 10*s.* 10*d.* for 11 Crowns and 7 Dollars; and at another Time I have 4 Crowns and 3 Dollars for 1*l.* 15*s.* each being of the same Value with the first. What is the Value of a Crown, and of a Dollar?

First

First 11 Crowns + 7 Dollars = 1090 *d.* } by the Question.
 Second 2 Crowns + 3 Dollars = 420 *d.* }

Then in order to find the Value of 1 Crown, you must cast off the Dollars by making them of the same Number ; Thus,

33 Crowns + 21 Dollars = 3270 *d.* the first multipl. with 3.
 28 Crowns + 21 Dollars = 2940 *d.* the second multipl. with 7.

Then 5 Crowns = 330 *d.* being the Difference.

Consequently 5) 330 (66 *d.* = 5 *s.* 6 *d.* is the Value of 1 Crown.
 And 4 Crowns = 264 *d.*

Then will 3 Dollars = 420 *d.* — 264 *d.* = 156 *d.*

Consequently 3) 156 (52 *d.* = 4 *s.* 4 *d.* the Value of 1 Dollar.

CHAP. IX.

Of Alligation.

WHEN it is required to mix several Sorts of Ingredients together ; as different sorts of Corn, Wines, Wool, Spices, or Metals ; or to compose Medicines, &c. the Method of proportioning such Mixtures, is called the *Rule of Alligation* ; and is divided into two Parts or Branches ; called *Medial* and *Alternate*.

Sect. I. Of Alligation Medial.

Alligation Medial, is that by which the Mean Rate or Price of any Mixture is found, when the particular Quantities of the Mixtures and Rates are given ; and is thus performed.

First find the Sum of all the Quantities proposed to be mixed ?
 And also the Sum of all their particular Rates.

Then the Proportion will be,

Rule { *As the Sum of all the Quantities : Is to the Sum of all their Rates :: So is any Part of the Mixture : To the Mean Rate or Price of that Part.*

Quest. 1. Suppose 15 Bushels of Wheat at 5 *s.* the Bushel, and 12 Bushels of Rye at 3 *s.* 6 *d.* the Bushel, were mixed together ;
 What

What is the Mean Rate or Price, it may be sold for a Bushel, without Loss or Gain?

This Question prepared as directed above, will stand

Thus $\left\{ \begin{array}{l} 15 \text{ Bushels of Wheat at } 5s. \text{ per Bushel, comes to } 900d. \\ 12 \text{ Bushels of Rye at } 3s. 6d. \text{ each, comes to } 504d. \end{array} \right.$
 $27 = \text{their Sum.} \quad \text{And their total Value} = 1404d.$

Then $27 \text{ Bushels} : 1404d. :: 1 \text{ Bushel} : 52d. = 4s. 4d.$ the Answer required.

Quest. 2. A Grocer mixeth 36 Pounds of Tobacco, worth $1s. 6d.$ a Pound, with 12 Pounds of another sort at $2s.$ a Pound, and 12 Pounds of a third sort at $1s. 10d.$ the Pound. How may he sell the Mixture *per Pound*?

	lb.	s.	d.		d.
First	$\left\{ \begin{array}{l} 36 . \text{ at } 1 . 6 \\ 12 . \text{ at } 2 . 0 \\ 12 . \text{ at } 1 . 10 \end{array} \right\}$			per Pound amounts to	$\left\{ \begin{array}{l} 648 \\ 288 \\ 264 \end{array} \right.$
					$60 = \text{the Number of Pounds. Their Value} = 1200$

Then $60 \text{ lb} : 1200d. :: 1 \text{ lb} : 20d. = 1s. 8d.$ the Answer required.

Quest. 3. A Vintner mixeth 31 Gallons and a half of *Malaga* Sack worth $7s. 6d.$ the Gallon; with 18 Gallons of Canary at $6s. 9d.$ the Gallon; 13 Gallons and a half of Sherry at $5s.$ the Gallon; and 27 Gallons of White Wine at $4s. 3d.$ the Gallon. It is required to find what one Gallon of this Mixture is worth.

	Gal.	s.	d.		Pence.
First	$\left\{ \begin{array}{l} 31\frac{1}{2} \text{ at } 7 . 6 \\ 18 \text{ at } 6 . 9 \\ 13\frac{1}{2} \text{ at } 5 . 0 \\ 27 \text{ at } 4 . 3 \end{array} \right\}$			per Gallon comes to	$\left\{ \begin{array}{l} 2835 \\ 1458 \\ 810 \\ 1377 \end{array} \right.$
					$90 = \text{the Number of Gall. Their Value} = 6480$

Then $90 : 6480 :: 1 : 72d. = 6s.$ the Rate or Price of one Gallon, as was required.

The Proof of all Operations in these sort of Mixtures, is done by comparing the Value of all the Mixture (being sold at the Mean Rate) with the total Value of all the particular Quantities, supposing they had been sold at their respective Rates unmixed; if those Sums are equal, the Work is true.

Sect.

Sect. 2. Of Allegation Alternate.

Alligation Alternate, is that by which the particular Quantities of every Ingredient concerned in any Mixture are found; when the particular Rates of every one of those Ingredients, and the mean Rate are given; being (as it were) the Converse to Allegation Medial; as will appear by the following Operations, which admit of three Cases.

Case I. The Particular Rates of any Ingredients proposed to be mixed, and the Mean Rate of the whole Mixture being given. To find how much of each Ingredient is requisite to compose the Mixture; when the whole Quantity, or any Part thereof, is not limited.

Quest. 1. How much Wheat at 5 s. the Bushel, and Rye at 3 s. 6 d. the Bushel, will compose a Mixture that may be sold for 4 s. 4 d. the Bushel?

Note, In all Questions of this Nature, it will be convenient to place the Mean Rate so, as that it may be easily compared with the Particular Rates, in order to find every one of their Differences from the Mean Rate, by Inspection only.

Thus, the Mean Rate = 52 d. $\left\{ \begin{array}{l} \text{Wheat } 60 \text{ d.} \\ \text{Rye } 42 \text{ d.} \end{array} \right.$

Then take the several Differences between the Mean Rate, and the Particular Rates; setting down those Differences alternately, and they will be the Quantities required.

Thus 52 $\left\{ \begin{array}{l} 60 \\ 42 \end{array} \right\} \left\{ \begin{array}{l} 10 = 52 - 42 \\ 8 = 60 - 52 \end{array} \right.$

That is $52 - 42 = 10$ for the Quantity of Wheat.

And $60 - 52 = 8$ for the Quantity of Rye, that will compose the Mixture required.

The Proof by *Alligation Medial*.

Add $\left\{ \begin{array}{l} 10 \text{ Bushels of Wheat at } 60 \text{ d. per Bushel} = 600 \text{ d.} \\ 8 \text{ Bushels of Rye at } 42 \text{ d. per Bushel} = 336 \text{ d.} \\ \hline 18 = \text{the Number of Bushels.} \qquad \qquad \qquad = 936 \text{ d.} \end{array} \right.$

Then $18 : 936 :: 1 : 52 \text{ d.} = 4 \text{ s. } 4 \text{ d.}$ the Mean Rate.

Note, Although 10 and 8 do answer the Question, as plainly appears by the Proof, yet they are not the only two Numbers; for this Question, and all others of this kind, will admit of various Answers, and all whole Numbers; for any two Numbers that are in the same Proportion to one another, as 10 is to 8, will as truly answer the Question.

Viz.

$$\text{Viz. } 10 : 8 :: \left\{ \begin{array}{l} 5 : 4 \\ 15 : 12 \\ 20 : 16 \\ 25 : 20 \end{array} \right\} \text{ \&c. ad infinitum.}$$

Quest. 2. A Grocer would mix three sorts of Tobacco together, viz. One Sort of 18 *d.* per lb. another Sort of 22 *d.* per lb. and a third Sort of 2 *s.* the lb. How much of each Sort must he take, that the whole Mixture may be sold for 20 *d.* the Pound?

Having set down the given Rates, as before, then find each of their Differences from the proposed Mean Rate, and place those Differences alternately. Thus,

$$\text{Mean Rate } 20 \left\{ \begin{array}{l} 18 \\ 22 \\ 24 \end{array} \right\} \left\{ \begin{array}{l} 4 + 2 = 24 - 20 \\ 2 = 20 - 18 \\ 2 = 20 - 18 \end{array} \right\}$$

These Differences, viz. 6 . 2 . 2 are the Quantities required.

$$\text{Proof. } \left\{ \begin{array}{l} 6 \text{ lb. of Tobacco at } 18 \text{ d.} \\ 2 \text{ lb. } \text{-----} \text{ at } 22 \text{ d.} \\ 2 \text{ lb. } \text{-----} \text{ at } 24 \text{ d.} \end{array} \right\} \text{ the Pound come to } \left\{ \begin{array}{l} 108 \text{ d.} \\ 44 \text{ d.} \\ 48 \text{ d.} \end{array} \right\}$$

10 = the Number of Pounds. Their Value = 200 *d.*

Then 10) 200 (20 the Mean Rate.

Or indeed any three Numbers that have the same Ratio to one another as 6 and 2 have, will answer the Question.

$$\text{That is, } 6 : 2 :: \left\{ \begin{array}{l} 9 : 3 \\ 12 : 4 \\ 15 : 5 \end{array} \right\} \text{ \&c.}$$

But if only one of the three given Rates had been greater than the Mean Rate; as suppose 14 *d.* per Pound, 18 *d.* per Pound, and 24 *d.* per Pound, and the Mean Rate 20 *d.* as before. Then their Differences must have been placed,

$$\text{Thus, } 20 \left\{ \begin{array}{l} 14 \\ 18 \\ 24 \end{array} \right\} \left\{ \begin{array}{l} 4 \\ 4 \\ 6 + 2 \end{array} \right\} \text{ \&c. as before.}$$

Quest. 3. A Vintner would make a Mixture of Malaga, worth 7 *s.* 6 *d.* per Gallon, with Canary at 6 *s.* 9 *d.* per Gallon, Sherry at 5 *s.* per Gallon, and White Wine at 4 *s.* 3 *d.* per Gallon; What Quantity of each Sort must he take, that the Mixture may be sold for 6 *s.* per Gallon?

In all Questions of this Kind, wherein it is required to mix four Things together, two of them having their Prices greater, and two lesser than the mean Rate: you must always alligate or
Q
compare

compare a greater and lesser Price with the mean Price, setting down their Differences alternately, as in the first *Example* of this *Section*.

$$\text{Thus, Mean Rate} = 72 d. \left\{ \begin{array}{l} \text{Malaga } 90 d. \\ \text{White } 51 d. \\ \text{Sherry } 60 d. \\ \text{Canary } 81 d. \end{array} \right\} \left\{ \begin{array}{l} 21 = 72 - 51 \\ 18 = 90 - 72 \\ 9 = 81 - 72 \\ 12 = 72 - 60 \end{array} \right.$$

Hence 21 Gallons of Malaga, 12 of Canary, 9 of Sherry, and 18 of White will compose the Mixture required.

$$\text{Or thus, } 72 \left\{ \begin{array}{l} \text{Malaga } 90 d. \\ \text{Sherry } 60 d. \\ \text{Canary } 81 d. \\ \text{White } 51 d. \end{array} \right\} \left\{ \begin{array}{l} 12 \text{ Malaga} \\ 18 \text{ Sherry} \\ 21 \text{ Canary} \\ 9 \text{ White} \end{array} \right\} \text{ will, \&c.}$$

Either of these Mixtures equally answer the Question, which may be easily tried as before in the last, &c.

Case II. The particular Rates of all the Ingredients proposed to be mixed, the Mean Rate of the whole Mixture, and any one of the Quantities to be mixed being given: Thence to find how much of every one of the other Ingredients is requisite to compose the Mixture.

Note, This is usually called *Alligation Partial*.

Quest. 4. How much Wheat at 5 s. the Bushel, must be mixed with 12 Bushels of Rye at 3 s. 6 d. a Bushel; that the whole Mixture may be sold for 4 s. 4 d. the Bushel?

In this Case you must set down all the particular Rates, with the Mean Rate, and find their Differences just as before; without any regard had to the Quantity given.

$$\text{Thus, Mean Rate } 52 d. \left\{ \begin{array}{l} \text{Wheat } 60 d. \\ \text{Rye } 42 d. \end{array} \right\} \left\{ \begin{array}{l} 10 \\ 8 \end{array} \right.$$

Then $\left\{ \begin{array}{l} \text{As the Quantity found by the Differences of the same} \\ \text{Name with the Quantity given: Is to the Quantity given ::} \\ \text{So is any of the other Quantities found by the Differences:} \\ \text{To the Quantity of it's Name.} \end{array} \right.$

Thus $8 : 82 :: 10 : 15$, the Quantity or Number of Bushels of Wheat required.

Quest. 5. How much Malaga at 7 s. 6 d. the Gallon, Sherry at 5 s. the Gallon, and White Wine at 4 s. 3 d. the Gallon, must be mixed with 18 Gallons of Canary at 6 s. 9 d. the Gallon; that the whole Mixture may be sold for 6 s. the Gallon?

*

The

The Terms being set down, &c. as before, will stand

Thus, Mean Rate 72 d. $\left\{ \begin{array}{l} \text{Malaga } 90 d. \\ \text{White } 51 d. \\ \text{Sherry } 60 d. \\ \text{Canary } 81 d. \end{array} \right\} \left\{ \begin{array}{l} 21 \\ 18 \\ 9 \\ 12 \end{array} \right.$

Then, as 12 : 18 :: $\left\{ \begin{array}{l} 21 : 31\frac{1}{2} \text{ Gallons of Malaga.} \\ 18 : 27 \text{ Gallons of White.} \\ 9 : 13\frac{1}{2} \text{ Gallons of Sherry.} \end{array} \right.$

That is, 31 $\frac{1}{2}$ Gallons of Malaga, 27 of White Wine, and 13 $\frac{1}{2}$ of Sherry, being mixed with 18 Gallons of Canary, will make the Mixture required.

Or thus, 72 $\left\{ \begin{array}{l} \text{Malaga } 90 \\ \text{Sherry } 60 \\ \text{Canary } 81 \\ \text{White } 51 \end{array} \right\} \left\{ \begin{array}{l} 12 \\ 18 \\ 21 \\ 9 \end{array} \right.$

Then, as 21 : 18 :: $\left\{ \begin{array}{l} 12 : 10\frac{6}{21} \text{ the Malaga.} \\ 18 : 15\frac{9}{21} \text{ the Sherry.} \\ 9 : 7\frac{15}{21} \text{ the White.} \end{array} \right\} \text{ \&c.}$

Gallons.			Pence.	
Proof.	$10\frac{6}{21}$ at 90 d.	$\left\{ \right.$ each $\left\{ \right.$	$925\frac{15}{21}$	$\left\{ \right.$
	$15\frac{9}{21}$ at 90 d.		$925\frac{1}{21}$	
	$7\frac{15}{21}$ at 51 d.		$393\frac{9}{21}$	
	18 at 81 d.		1458	
Sum	$51\frac{2}{21}$		Value =	$3702\frac{18}{21}$

Then $51\frac{2}{21} \times 3702\frac{18}{21}$ (72 d. = 6 s. the Mean Rate.

Therefore the Quantities are as truly assigned here, as in the last Work.

Case III. The particular Rates of all the Ingredients proposed to be mixed; and the Sum of all their Quantities with the Mean Rate of that Sum being given; to find the particular Quantities of the Mixture.

This is called *Alligation Total*, and is thus performed.

Set down all the particular Rates, with the Mean Rate, and find their Differences, as before: add together all the Differences into one Sum;

Then $\left\{ \begin{array}{l} \text{As the Sum of all the Differences : Is to the Sum of all} \\ \text{the Quantities given :: So is every particular Difference :} \\ \text{To it's particular Quantity.} \end{array} \right.$

Quest. 6. Let it be required to mix Wheat at 5 s. the Bushel, with Rye at 3 s. 6 d. the Bushel; so that the whole Quantity may be 27 Bushels, to be sold for 4 s. 4 d. a Bushel; what Quantity of each must be taken to make up the Mixture?

Q 2

Mean

$$\text{Mean Rate } 52 \left\{ \begin{array}{l} \text{Wheat } 60 d. \\ \text{Rye } 42 d. \end{array} \right\} \left\{ \begin{array}{l} 10 \\ 8 \end{array} \right\} \\ \hline 18 = \text{their Sum.}$$

$$\text{Then } 18 : 27 :: \left\{ \begin{array}{l} 10 : 15 \\ 8 : 12 \end{array} \right\} \text{ the Quantities required.}$$

Question 7. Suppose it were required to mix Malaga at 7s. 6d. the Gallon, with Canary at 6s. 9d. the Gallon; Sherry at 5s. the Gallon, and White Wine at 4s. 3d. the Gallon; so that the whole Mixture may be 90 Gallons; to be sold for 6s. the Gallon: How much of each sort will compose that Mixture?

$$\text{Mean Rate} = 72 d. \left\{ \begin{array}{l} \text{Malaga } 90 \\ \text{White } 51 \\ \text{Canary } 81 \\ \text{Sherry } 60 \end{array} \right\} \left\{ \begin{array}{l} 21 \\ 18 \\ 9 \\ 12 \end{array} \right\} \\ \hline 60 = \text{their Sum.}$$

$$\text{Then } 60 : 90 :: \left\{ \begin{array}{l} 21 : 31 \frac{1}{2} \\ 18 : 27 \\ 9 : 13 \frac{1}{2} \\ 12 : 18 \end{array} \right\} \text{ the Gallons of } \left\{ \begin{array}{l} \text{Malaga.} \\ \text{White Wine.} \\ \text{Sherry.} \\ \text{Canary.} \end{array} \right.$$

$$\text{Or thus, } 72 \left\{ \begin{array}{l} \text{Malaga } 90 \\ \text{Sherry } 60 \\ \text{Canary } 81 \\ \text{White } 51 \end{array} \right\} \left\{ \begin{array}{l} 12 \\ 18 \\ 21 \\ 9 \end{array} \right\} \\ \hline 60 = \text{their Sum.}$$

$$\text{Then } 60 : 90 :: \left\{ \begin{array}{l} 12 : 18 \\ 28 : 27 \\ 21 : 31 \frac{1}{2} \\ 9 : 13 \frac{1}{2} \end{array} \right\} \text{ Gallons of } \left\{ \begin{array}{l} \text{Malaga.} \\ \text{Sherry.} \\ \text{Canary.} \\ \text{White Wine.} \end{array} \right.$$

Either of these Ways do equally answer the Question, as may be easily tried by *Alligation Medial*. As before, &c.

Note, *The Work of these Proportions may be much shortened (especially when there are many Ingredients to be mixed) if you observe the same Method as was proposed in the Rule of Fellowship, page 99, &c.*

I have made Use of the very same *Examples* both in *Alligation Medial*, and *Alternate*, throughout the three *Cases*; being, as I presume, much better than if they had been different ones; because the Learner may (if he considers them a little) easily perceive, not only the Difference between the two Rules, but also wherein the

the chief Difference of each Case in the *Alternate Rule* depends, &c. Not but that I could have inserted many various *Examples*, as also the Manner of composing Medicines, &c. which, for Brevity sake, I have omitted, and refer those that desire to see into that Business to Sir *Jonas More's Arithmetick*, wherein he will find it largely handled. And so I shall conclude with *Alligation Alternate*, which altho' it gives true Answers to Questions of that Kind, with some little Variety, according as the Ingredients are more or less in Number; as appears by the foregoing *Examples*; yet it will not give all the Answers such Questions are capable of, nor perhaps those which suit best with the present Occasion: Nor can this Imperfection be remedied by common *Arithmetick*; but by an *Algebraick* way of arguing it may; whereby all the possible Answers to any Question may be clearly and easily discovered; as shall be shewed further on in the Second Part.

C H A P. X.

Of Metals and their Specifick Gravities, &c.

Sect. 1. *Of Gold and Silver.*

PURE Gold, free from Mixture with other Metals, usually called Fine Gold, is of such a Nature and Purity that it will endure the Fire without wasting, although it be kept continually melted: And therefore some of the ancient Philosophers have supposed the Sun to be a Globe of liquid or melted Gold.

Silver having not the Purity of Gold, will not endure the Fire like it: Yet Fine Silver will waste but a very little by being in the Fire any reasonable time; whereas Copper, Tin, Lead, &c. will not only waste, but may be calcined or burnt to a Powder.

Both Gold and Silver in their Purity, are so very flexible or soft (like new Lead, &c.) that they are not so useful either in Coin, or otherwise (except to beat in Leaf-Gold or Silver) as when they are allay'd, or mixed and hardened with Copper or Brass. And altho' most Places differ more or less in the Quantity of such Allay, yet in *England* it is generally agreed on, that,

Standard

Standard for Gold.

22 Carats of Fine Gold, and 2 Carats of Copper, being melted together, shall be esteemed the true Standard for Gold Coin, &c. (*The French and Spanish Gold being very near of the same Standard.*) That is, if any Quantity or Weight of Fine Gold, be divided into Twenty-four equal Parts, and 22 of those Parts be mixed with 2 of the like Parts of Copper; that Mixture is called Standard Gold.

Whence you may observe, that a Carat is not any certain Quantity or Weight, but $\frac{1}{24}$ part of any Quantity or Weight; and the *Minters* and *Goldsmiths* divide it into 4 equal Parts, which they call Grains of a Carat; also they subdivide one of those Grains, into Halves, Quarters, &c.

Standard for Silver.

Eleven Ounces and Two Penny-weight of Fine Silver, and Eighteen Penny-weight of Copper being melted together, is esteemed the true Standard for Silver Coin, called Sterling Silver. And so in Proportion for a greater or lesser Quantity; which is a less Proportion of Allay for Silver, than the other is for Gold.

Note, When either Silver or Gold is finer than Standard, it is called Better; if coarser, it is called Worse; and that Betterness or Worseness, is reckoned by Carats and Grains of a Carat in Gold, and by Penny-weights in Silver; and is thus discovered: The *Goldsmiths* or *Refiners*, &c. take a small Quantity of such Gold as they intend to try (which they call making an *Affay*) and weigh it very exactly, then they put it into a Crucible, and melt it in a strong Fire, so long, that if there be any Copper, or other Allay mixt with it, that Allay may be consumed or burnt away: When it is cold they weigh it very exactly again, and if it have lost nothing of it's first Weight, they conclude it is Fine Gold, but if the Loss be $\frac{1}{24}$ Part, they call it 23 Carats Fine, or one Carat better than Standard: If it have lost $\frac{2}{24}$ Parts it is 22 Carats fine, or Standard: If $\frac{3}{24}$ Parts, it is said to be 21 Carats fine, or rather one Carat worse than Standard, and so in Proportion as it happens to be better or worse.

In the same Manner they make their Affay on Silver, only they compute it's Loss by Penny-weights, &c.

The Author of the *Present State of England*, mentioned before (*page 32.*) says,

‘ That

‘ That the *English* Coin may want neither the Purity nor Weight required, it is most wisely and carefully provided, that once every Year the chief Officers of the *Mint* appear before the Lords of the Council in the *Star-Chamber* at *Westminster*, with some Pieces of all sorts of Monies coined the foregoing Year, taken at adventure out of the *Mint*, and kept under several Locks, by several Persons, ’till that Appearance, and then by a Jury of 24 able *Goldsmiths*, in the Presence of the said Lords, every Piece is most exactly weighed and assay’d.’

This if it were constantly practised would keep our Coin to it’s true Standard, &c.

Many pretty Questions may be started concerning the Fineness of Gold and Silver, &c.

E X A M P L E 1.

If an Ingot of Silver weighing 787 Oz. 14 Pwt. 6 Grains, be 11 Oz. 6 Pwt. fine; How much fine Silver is there in it, and what amounts it to, at 5 s. 1½ d. the Ounce?

This Ingot is better than Standard by 4 Pwt. For 11 Oz. 2 Pwt. = 222 Pwt. the fine Silver in 12 Oz. of Standard. But 11 Oz. 6 Pwt. = 226 Pwt. the fine Silver in 12 Oz. according to the Question.

First 787 Oz. 14 Pwt. 6 Grains = 378102 Grains.

And 12 Ounces = 240 Pwt.

Then, As 240 : 226 :: 378102 : $356046\frac{1}{20}$ = 741 Oz. 15 Pwt. 6 $\frac{1}{20}$ Grains the fine Silver in that Ingot.

Which at 5 s. 1½ d. the Ounce, amounts to 190 l. 1 s. 6 d. and near a Half-penny.

E X A M P L E 2.

If an Ingot of Gold weighing 115 Oz. 13 Pwt. 18 Grains; be $\frac{1}{4}$ of a Grain worse than Standard: How much Standard Gold is there in it, and what comes it to at 3 l. 11 s. an Ounce?

First 115 Oz. 13 Pwt. 18 Grains = 55530 Grains Troy.

Then 24) 55530 (2313,75 = a Carat of that Quantity.

And 4) 2313,75 (578,4375 = a Grain of that Carat.

Consequently 4) 578,4375 (144,609375 = $\frac{1}{4}$ of a Grain.

Again, $2313,75 \times 22 = 50902,5$ ought to be the fine Gold in that Ingot, if it had been Standard:

But

But $50902,5 - 144,609375 = 50757,890625$ is the Quantity of fine Gold according to the Question. Therefore $50902,5 : 50757,890625 :: 55530 : 55372,244 \text{ \&c. Grains} = 115 \text{ Oz. } 7 \text{ Pwt. } 4,244 \text{ \&c. Grains Troy}$, being the Quantity of Standard Gold in that Ingot, as was required.

Next for the Value of it at $3 \text{ l. } 11 \text{ s. per Ounce}$; $1 \text{ Oz.} = 480 \text{ Grains}$; and $3 \text{ l. } 11 \text{ s.} = 71 \text{ s.}$ Consequently $480 : 71 :: 55372,244 \text{ \&c.} : 8190,4777 \text{ \&c.} = 409 \text{ l. } 10 \text{ s. } 5 \frac{3}{4} \text{ d.}$ very near; being the Value of that Ingot, as was required.

Or the last Question may be otherwise wrought thus; $115 \text{ Oz. } 13 \text{ Pwt. } 18 \text{ Grains} = 115,6875$. And $\frac{1}{4}$ of a Grain of a Carat is $\frac{1}{16}$ (*viz.* the $\frac{1}{4}$ of $\frac{1}{4}$) Then $22 - \frac{1}{16} = 21 \frac{15}{16} = 21,9375$. Consequently $22 : 21,9375 :: 115,6875 : 115,358842 \text{ \&c.} = 115 \text{ Oz. } 7 \text{ Pwt. } 4,244 \text{ Grains, \&c.}$ as before.

Next for the Value; as $1 : 3,55 :: 115,358842 : 409,523,889 = 409 \text{ l. } 10 \text{ s. } 5 \frac{3}{4} \text{ d.}$ very near: as before.

Sect. 2. *The Specifick Gravity of Metals, &c.*

I Take an Enquiry made about the different Gravities, or Weights of Metals, and other Bodies, to be (not only a Work of Curiosity, but also) of very good Use upon many Occasions. Therefore several Authors have given us such Proportions, or Difference of their Weights, as they are said to have one to another; supposing every one of them to be of the same Magnitude or Bigness. Some of which I shall here insert.

1. *Henry Van Etten*, in his *Mathematical Recreations*, printed Anno 1633, sets down the Proportion of their Weights thus; Gold 1875. Lead 1165. Silver 1040. Copper 910. Iron 810. Tin 750. Water 100.

2. One *Alsted*, in his *Encyclopædia*, printed Anno 1649, hath them thus: Gold 1875. Quicksilver 1500. Lead 1165. Silver 1040. Copper 910. Iron 806. Tin 750. Honey 150. Water 100. Oil 90. These seem to be taken from those of *Van Etten's*, with some Additions only.

3. The ingenious Mr *Oughtred*, in his *Circles of Proportions*, printed Anno 1660, hath their Proportions (according to the Experiments of one *Marinus Ghetaldi*, in his Tract called *Archimedes Promotus*) thus: Gold 3990. Quicksilver 2850. Lead 2415. Silver 2170. Brass 1890. Iron 1680. Tin 1554.

4. In

4. In the Philosophical Transactions, (*Number 169 and 199*) there is an Account of a great many Experiments of this Kind; from whence I collected these following, *viz.* Gold 18888. Mercury 14019. Lead 11343. Silver 11087. Copper 8843. Hammered Brass 8349. Cast Brass 8100. Steel 7852. Iron 7643. Tin 7321. Pump-water 1000.

These last Proportions being approved of and published by Order of the *Royal Society* seem to be unquestionably true: Nevertheless, because they differ so much from the beforementioned (*and those from one another*) I have for my own Satisfaction made several Experiments of that Kind: And have (*I presume*) obtained the Proportions of Weight that one Body bears to another of the same Bulk or Magnitude, as nicely as the Nature of such Matter, which may be contracted or brought into a lesser Body (*viz.* either by Drying, or Hammering, or otherwise) will admit of; which are as follow:

		Ounces Troy.		Ounces Avoird.
A Cubick Inch of	Fine Gold, is - -	10,359273	=	11,365602
	Standard Gold - -	9,962625	=	10,930422
	Quicksilver - - -	7,384411	=	8,101753
	Lead - - - - -	5,984010	=	6,553885
	Fine Silver - - -	5,850035	=	6,418324
	Standard Silver - -	5,556769	=	6,096569
	Rose Copper - - -	4,747121	=	5,208369
	Plate Brass - - -	4,404273	=	4,832116
	Cast Brass - - -	4,272409	=	4,630300
	Steel - - - - -	4,142127	=	4,544505
	Common Iron - - -	4,031361	=	4,422979
	Block Tin - - -	3,861519	=	4,236638
	Fine Marble - - -	1,429411	=	1,568859
	Common Glass - - -	1,360841	=	1,493037
	Alabaster - - -	0,988456	=	1,084477
	Dry Ivory - - -	0,962083	=	1,055542
	Dry Box-wood - -	0,543282	=	0,596057
	Sea Water - - -	0,542742	=	0,594894
	Common clear Water	0,527458	=	0,578697
	Red Wine - - -	0,523766	=	0,574646
	Proof Spirits of Brandy	0,489268	=	0,536796
	Sound Dry Oak - -	0,489008	=	0,536569
	Linseed Oil - - -	0,491591	=	0,539345
	Oil Olive - - -	0,481569	=	0,528350

In this Table you have the Specifick Gravity or Weight of a Cubic Inch, of various sorts of Bodies, both in *Troy* Ounces and *Avoirdupois* Ounces, and Decimal Parts of an Ounce, which I can assure you required more Charge, Care, and Trouble, to find out nicely, than I was at first aware of.

Now from hence it will be easy to determine the Weight of any proposed Quantity, of the same Matter and Kind with those in the Table; it's Solid Content being given in Cubic Inches. For it is plain, that if the Number of Cubic Inches contained in any given Quantity, be multiplied with the tabular Weight of one Inch, (*of the same Kind of Matter*) the Product will be the Weight of that Quantity in Ounces, &c.

E X A M P L E.

Suppose it were required to find the Weight of a Piece of Marble, containing three Solid Feet, and 40 Cubic Inches.

First $1728 \times 3 = 5184$ the Cubic Inches in 3 Solid Feet.

And $5184 + 40 = 5224$ the Number of Cubic Inches in the Piece of Marble.

Then $5224 \times 1,429411 = 7410,066624$ Ounces *Troy*.

Or $5224 \times 1,568859 = 8195,719416$ Ounces *Avoirdupois*.

The Weight of that Piece of Marble, in Ounces, &c. which is easily brought into Pounds, &c. The like for any of the rest.

The Converse of this Work is as easy; *viz.* if the Weight of any proposed Quantity be given, thence to find the Solid Content of that Quantity in Cubic Inches, &c.

Thus, divide the given Weight of the proposed Quantity (*it being first reduced into Ounces, &c.*), by the tabular Weight of one Inch (*of the same Kind of Matter*), and the Quotient will be the Number of Cubic Inches contained in that Quantity.

Note, If you would find what Weight any Quantity of those Bodies mentioned in the Table will have, when it is immersed or put into Water, you must subtract the Weight of an equal Quantity of Water (with that of the Body), from the Weight of the proposed Body (if it be heavier than Water), and there will remain the Weight required. As for Instance,

A Cubic Inch of Lead	= 5,984010	} Ounces <i>Troy</i> , &c.
A Cubic Inch of Sea Water	= 0,542742	
their Difference is = 5,441268		the Weight of a Cubic Inch of Lead in the Water, &c.

C H A P. XI.

Evolution, or Extracting the Roots out of all Single Powers; by one Geometrical Method.

S E C T. I.

E*volution* is the Unravelling, or as it were the Unfolding and Resolving any proposed Power or Number, into the same Parts of which it was composed, or supposed to be made up. Now in order to perform that, it will be convenient to consider how those Powers are composed, &c.

A Square Number is that which is equally equal; or which is contained under two equal Numbers. *Euclid. 7. Def. 18.* Thus the Square Number 4 is composed of the two equal Numbers 2 and 2. *viz.* $2 \times 2 = 4$. Or the Square Number 9 is composed of the two equal Numbers 3 and 3. *viz.* $3 \times 3 = 9$: according to *Euclid.* That is, if any Number be multiplied into itself; that Product is called a Square Number.

A Cube is that Number which is equally equal, or which is contained under three equal Numbers. *Eucl. 7. Def. 19.* Thus the Cube Number 8 is composed of the three equal Numbers 2 and 2 and 2. *viz.* $2 \times 2 \times 2 = 8$, &c. That is, if any Number be multiplied into itself, and that Product be multiplied with the same Number; the second Product is called a Cube Number.

These two, *viz.* the Square and Cube Numbers, borrow their Names from *Geometrical Extensions* or Figures; as from the three Signal Quantities mentioned in *page 2.* That is, a *Root* is represented by a **Line** or **Side**, having but one Dimension, *viz.* that of **Length** only. The Square is a Plane or Figure of two Dimensions, having equal **Length** and **Breadth**. The Cube is a Solid Body of three Dimensions; having equal **Length**, **Breadth**, and **Thickness**: But beyond these three, Nature proceeds not, as to *Local Extension*. That is, the Nature of Place or Space, admits no Room for other ways of Extension, than Length, Breadth, and Thickness. Neither is it possible to form, or compose any Figure or Body beyond that of a Solid.

And therefore all the superior Powers above the Cube or third Power; as the *Biquadrat* or fourth Power, the *Sur-solid* or fifth Power, &c. are best explained and understood by a Rank or Series of Numbers in *Geometrical Proportion*. For Instance: Suppose any Rank of *Geometrical Proportionals*, whose first Term and Ratio are the same; and to them let there be assigned a Series

of Numbers in *Arithmetical Progression*, beginning with an Unit or 1, whose common Difference is also 1, as in page 79.

Thus, $\left\{ \begin{array}{l} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \text{ Indices.} \\ 2 \cdot 4 \cdot 8 \cdot 16 \cdot 32 \cdot 64 \cdot 128 \text{ \&c. in } \div \end{array} \right.$

Then are those Numbers in $\div$ produced by a continued Multiplication of the first Term or Root into itself; and those in *Arithmetical Progression* or **Indices**, do shew what Degree or Power each Term in the *Geometrical Proportion* is of. For Example; In this Series of $\div 2$ is both the first Term or Root, and common Ratio of the Series. Then $2 \times 2 = 4$ the second Term or Square; and $2 \times 2 \times 2 = 8$, or $4 \times 2 = 8$, the Cube or third Term; $2 \times 2 \times 2 \times 2 = 16$, or $8 \times 2 = 16$ the fourth Term or Biquadrat. And so on for the rest.

Note, This is called **Involution**, viz. When any Number is drawn into itself, and afterwards into that Product, &c. it is said to be so often involved into itself; and the Indices are the Exponents of their respective Powers so involved.

And according to these Involutions, is formed the following Table of Powers; wherein the Root is only one single Figure.

Root, or single Side.	Square, or second Power.	Cube, or the third Power.	Biquadrat, or Square squared; being the fourth Power.	Sur-solid, or the fifth Power.	Square cubed, or Cube squared; the sixth Power.	The second Sur-solid, or seventh Power.	The Biquadrat squared, or the eighth Power.	The Cube cubed, or the ninth Power, &c.
Index (1)	Index (2)	Index (3)	Index (4)	Index (5)	Index (6)	Index (7)	Index (8)	Index (9)
1	1	1	1	1	1	1	1	1
2	4	8	16	32	64	128	256	512
3	9	27	81	243	729	2187	6561	19683
4	16	64	256	1024	4096	16384	65536	262144
5	25	125	625	3125	15625	78125	390625	1953125
6	36	216	1296	7776	46656	279936	1679616	10077696
7	49	343	2401	16807	117649	823543	5764801	40353807
8	64	512	4096	32768	262144	2097152	16777216	134217728
9	81	729	6561	59049	531441	4782969	43046721	387420489

This Table plainly shews (by Inspection) any Power (under the Tenth) of all the nine Figures; and from thence may be taken the nearest Root of any Square, Cube, Biquadrat, &c. of any Number whose Root or Side is a single Figure.

But

But if the Root consists of two, three, or more places of Figures, then it must be found by piece-meal, or Figure after Figure, at several Operations.

The Extraction of all Roots, above the Square (*viz.* of the Cube, Biquadrat, Sur-solid, &c.) hath heretofore been a very tedious and troublesome Piece of Work: All which is now very much shortened, and rendered easy, as will appear further on.

When any Number is proposed to have it's Root extracted, the first Work is to prepare it, by Points set over (or under) their proper Figures; according as the given Power, whose Root is sought doth require; and that is done by considering the Index of the given Power, which for the Square is 2, for the Cube 3, for the Biquadrat is 4, &c. (as in the precedent Table) Then allow so many Places of Figures in the given Power, for each single Figure of the Root, as it's Index denotes; always beginning those Points over the Place of Unity, and ascend towards the Left-Hand if the given Number be Integers, and descend towards the Right-Hand in Decimal Parts. As in these following.

Suppose any given Number; as 75640387246 which I shall all along hereafter call the Resolvend.

Then if it be required to extract any of the following Roots, it must be pointed (according to the forementioned Consideration) in this manner:

Viz. For the	{	Square Root	Thus	75640387246
		Cube Root		75640387246
		Biquadrat Root		75640387246
		Sur-solid Root		75640387246

Or suppose the Number to be 0,674035982

Then for the	{	Square Root	Thus	0,6740359820
		Cube Root		0,674035982
		Biquadrat Root		0,674035982000

Now the Reason of pointing the given Resolvend in this manner; *viz.* the allowing two Figures in the Square; three Figures in the Cube, and four Figures in the Biquadrat, &c. for one Figure in the Root, may be made evident several ways; but I think it is easily conceived from the Table of single Powers, wherein you may observe that all the Powers of the Figure 9 (which

(which is but a single Figure) have the same Number of Places of Figures, as the Index of those Powers denotes: Therefore so many Places of Figures must be taken or assigned for every single Figure in the Root. Consequently by these Points is known how many Places of Figures there will be in the Root, *viz.* So many Points as there are, so many Figures there must be in the Root, and whether they must be Integers or Decimal Parts, is easily determined by the respective Places of the Points.

Sect. 2. To Extract the Square Root.

AND first how to extract the Square Root, according to the common Method.

Having pointed the given Resolvend into Periods of two Figures as before directed; then by the Table of Powers (or otherwise) find the greatest Square that is contained in the first Period towards the Left-Hand (setting down it's Root, like a Quotient Figure in *Division*) and subtract that Square out of the said Period of the Resolvend: To the Remainder bring down the next Period of Figures, for a Dividend, and double the Root of the first Square for a Divisor; enquiring how oft it may be had in that Dividend, so as when the Quotient Figure is annexed to the Divisor, and that increased Divisor multiplied with the same Quotient Figure, the Product may be the greatest Number that can be taken out of that Dividend; which subtract from the said Dividend, and to the Remainder bring down the next Period of Figures, for another new Dividend: Then see how often the last increased Divisor, can be had in the new Dividend (*with the same Caution as before, viz.*), so as that the Quotient Figure being annexed to the Divisor, and that increased Divisor multiplied with the same Quotient Figure, their Product may be the greatest Number that can be subtracted from the new Dividend. (As before) And so proceed on from Period to Period (*viz.* from Point to Point) in the very same Manner, until all be finished.

An Example or two being well observed will render the Work of forming the new Divisors, &c. more plain and easy than can be expressed in a Multitude of Words.

Example 1. Let it be required to extract the Square Root out of 572199960721. This Resolvend being prepared or pointed as before directed, will stand

Thus,

Thus, 572199960721 (756439 the Root.
49 = the greatest Square in 57.

1. Divisor	145)	821	
	5	725	= 145 × 5
2. Divisor	1506)	9699	
	6	9036	= 1506 × 6
3. Divisor	15124)	66396	
	4	60496	= 15124 × 4
4. Divisor	151283)	590007	
	3	453849	= 151283 × 3
5. Divisor	1512869)	13615821	
	9	13615821	= 1512869 × 9

Proof $756439 \times 756439 = 572199960721$ the Resolvend.

Example 2. What is the Square Root of 1850701,764025?

Operation $1850701,764025$ (1360,405

	1	
23)	85	
3	69	
266)	1607	
6	1596	
17204)	1101,76	
4	1088 16	
1720805)	13 604025	
5	13 604025	
	(0)	

Hence 1360,405 is the Root required.

Ex. 3. What is the Square Root of 0,06076225 Decimal Parts?

Operation $0,06076225$ (0,2465 the Root required.

	.04 = .2 × .2	
,44)	207	
4	176	
,486)	3162	
6	2916	
,4925)	24625	
5	24625	
	(0)	

Proof { $0,2465 \times 0,2465 =$
0,06076225 the Resolvend.

What

What is here done in whole Numbers, mixed Numbers, and Decimals, may also be done in Vulgar Fractions; if you first change the given Fraction into Decimals. (As in *Seet. 5. p. 68.*)

Example 4. Let it be required to extract the Square Root of $\frac{16}{25}$.
First $\frac{16}{25} = 0,64$

Then $0,64$ (*,8 the Root required.*)

$$\begin{array}{r} .64 \\ \hline (0) \end{array}$$

In these four Examples the Resolvend hath been a perfect Square; and therefore the Root hath been extracted without leaving any Remainder: But it very often happens that the Resolvend is not a true Figurate Number, according to the proposed Power. That is, it is not a perfect Square, Cube, Biquadrat, &c. and then something will remain after the Extraction hath been made throughout all the Points. Such Numbers are called **Surd** Numbers, and their Roots can never be truly found, but will become a continued Series, *ad infinitum*: If to the Remainder there be still annexed Cyphers according as the proposed Power requires, *viz.* by two's in the Square; three's in the Cube, four's in the Biquadrat, &c. And the Operations continued on as before.

Example 5. Suppose it were required to extract the Square Root of 6968.

Operation	6968 (83,4745, &c.
	64
163)	568
3	489
1664)	79,00
4	66 56
16687)	12 4400
7	11 6809
166944)	759100
4	667776
1669485)	9132400
5	8347425
1669490)	784975 &c.

Then the Root of any Surd Number may be continued on to what Exactness you please, but cannot be truly found.

In my *Compendium of Algebra*, Chap. 9. I have proposed another Way of extracting the Square Root, and there given Examples of the Work; which to avoid Prolixity is thus;

Having

Having pointed the given Resolvend, and taken the greatest Square to the first Point from it, as before. Then divide the Remainder of the whole Resolvend by 2 (that is, half it) and point it a-new. (This I call a new Dividend) Then make the Root of the first Square a Divisor, inquiring how oft it may be found in the new Dividend to the next figure forward, reserving that Figure under the next Point for the half Square of the Quotient Figure. Which being found, multiply the Divisor with it, adding to that Product the Tens of the half Square if there be any, as in plain Division. Then annex the Quotient Figure to the last Divisor for a new Divisor, with which proceed in all Respects as with the last Divisor; and so on until all be finished.

Example 6. What is the Square Root of 2990667969

Operation $\begin{array}{r} \cdot \cdot \cdot \cdot \cdot \cdot \\ 2990667969 \\ - 25 \end{array}$ (5 The first single Root
2) 490667969 The Remainder to be divided by 2.

First Root 5) $\begin{array}{r} \cdot \cdot \cdot \cdot \cdot \cdot \\ 245333984,5 \end{array}$ (54687
+ 4 $208 = 56 \times 4 : + \frac{1}{2}$ the Square of 4, viz. $\frac{16}{2} = 8$.
Divisor 54) 3733
+ 6 $3258 = 54 \times 6 : + \frac{1}{2}$ the Square of 6.
Divisor 546) 47539
+ 8 $43712 = 546 \times 8 : + \frac{1}{2}$ the Square of 8.
Divisor 5468) 382784,5
+ 7 $382784,5 = 5468 \times 7 : + \frac{1}{2}$ the Square of 7.
(0)

Hence the Root is found to be 54687, as was required.

All the Difficulty in this Method is only the true placing of the half Square of the Quotient Figure, when it happens to be an odd Number: In that Case you must bring down one Figure more of the Dividend; viz. of the next Period; under which, place the odd 5 that will always arise from the half Square of an odd Number: As 7 whose Square is 49; the Half of which is 24,5 to be placed as in the last Operation of this Example.

N. B. When the Number of Figures in the Root of any Surd Number is limited; you need not proceed in extracting the whole Root as before; but only to one Figure more than half the designed Number of Figures; for the rest may be obtained by plain Division only.

Example 7. Suppose it were required to extract the Square Root of 7 (a Surd Number) to have 12 Places of Figures in it.

	7 (2,645751	First part of the Root.
	4	
Remainder	3	
2)	1,50 = Half the Remainder.	
+ ,6	1,38 = 2 × ,6 : + $\frac{1}{2}$ the Square of 0,6 = 0,18	
2,6)	1200	
+ ,04	1048	
2,64)	152000	
+ ,005	132125	
2,645)	1987500	
+ ,0007	1851745	
2,6457)	13575500	
+ ,00005	13228625	
2,64575)	34687500	
+ ,000001	26457505	
2,645751	8229995	

Having thus got 7 of the 12 Figures required in the Root; the rest may be easily found by the contract Way of Division proposed in *page 68*.

Thus 2,645751) 8229995	(2,64575131106
.... 7937253	
292742	
264575	
28167	
26457	
1710	
1097	
(13)	

Hence I find the Root of 7 to be 2,64575131106, as was required.

Thus you have two ways of extracting the Square Root, either of which may be practised as every one likes best.

Sect. 3. To extract the Cube Root.

THE Method I shall here propose for extracting the Cube Root admits of two Cases; both which are to be very well observed.

Having pointed the given Resolvend, (as before directed) viz. into Periods of three Figures; then seek a Cube Number by the Table of Powers (or otherwise) that comes nearest to the first Period of the Resolvend, whether it be greater or less than that Period.

Case 1. If the Cube Number so taken, be less than the first Period of the Resolvend, call it's Root **Less** than **Just**: And subtract that Cube from the first Period of the Resolvend.

Case 2. But if that Cube be greater than the first Period of the Resolvend, call it's Root **More** than **Just**: And subtract the Resolvend from that Cube, annexing Cyphers to it, that so Subtraction may be made.

To the first Root, whether it be less or more than just, annex so many Cyphers as there are remaining Points over the whole Numbers of the Resolvend, and multiply it with 3: Then making that Product a Divisor, by which you must divide the Difference between the Resolvend and the foresaid Cube; that Quotient will be the Resolvend depressed to a Square, and therefore must be pointed as such, viz. into Periods of two Figures each. That being done, make the first Root (without those Cyphers that were annexed to it) a Divisor, enquiring how oft it may be found in the first Period of the new Resolvend (as before in extracting the Square Root) with this Consideration, that if the Root (now a Divisor) be less than just, as in *Case 1.* you must annex the Quotient Figure to it, and then multiply the Root so increased, into the said Quotient Figure; setting down the Unit's Place of their Product under the pointed Figure of that Period, subtracting it, as in Division. And so on from one Period to another, as before.

But if the said Root (now a Divisor) be more than just, as in *Case 2.* Then you must subtract the Quotient Figure from a Cypher annexed, or supposed to be annexed, to the said Divisor; multiplying the Root so decreased into the Quotient Figure; setting down their Product as before, &c. An *Example* or two in each *Case* will render the Work plain and easy.

Note, Each Quotient Figure ought always to be twice added to the Divisor, if the Tabular Cube was taken less than just, or twice subtracted from it, if greater; viz. once before you multiply by it, and once with the next Quotient Figure: as will be shewn in the following Examples; which are therefore more exact and concise than as done by the Author in all the former Editions of his Work.

Ex. 1. What is the Cube Root of 146363183 the given Resolvend, to be pointed thus 146[.]363[.]183[.] (the first Root, less than just.

125 = the nearest Cube to 146

500 × 3 = 1500) 21363183 (14242, 12 new Resolvend

First Root 5)

$$\begin{array}{r} + 2 \\ 1 \text{ Divisor } 52) \\ + 27 \\ 2 \text{ Divisor } 547) \end{array}$$

14242, 12 (527 the Root required.

$$\begin{array}{r} 104 \\ 3842 \\ 3829 \end{array}$$

13 the Remainder to be rejected.

Here the Root 527 is the true Root at the first Operation, as may be easily tried by involving it.

That is $527 \times 527 \times 527 = 146363183$ the given Resolvend. But if it had not been the true Root, then every thing that hath been here done must have been repeated; only instead of the first single Root (*viz.* 5) you must have taken the increased Root (*viz.* 527) and this I call a second Operation; which would increase the last Root to nine Places of Figures; *viz.* every Operation triples the Number of Places in the last Root; as will appear further on.

N. B. It often happens that four, five, and sometimes more Places of Figures may be taken into the Root: especially when the second Place proves to be a Cypher. That is, when the first Cube comes very near to the first Period of the Resolvend.

E X A M P L E 2.

What is the Cube Root of 67507824239 (4000 Root less than First nearest Cube = 64 [just.

Root $4000 \times 3 = 12000$) 3507824239 (292318, 68

First Root 4)

$$\begin{array}{r} + 07 \\ 1 \text{ Divisor } 407) \\ + 071 \\ 2 \text{ Divisor } 4141) \\ + 1,7 \\ 3 \text{ Divisor } 4142,7) \\ + ,79 \\ 4 \text{ Divisor } 4143,49) \end{array} \quad \begin{array}{r} \dots \\ 292318,68 \text{ (4071,79} \\ 2849 \\ 7418 \\ 4141 \\ 3277,68 \\ 2899,87 \\ 377,79 \text{ \&c.} \end{array}$$

In this Example I have taken six Figures into the Roots, because the second Place proved to be a Cypher. And in these six the

99200)			
+ 06			
99206)	6321068181	(9920000000 the Root assumed,	
+ 63	595236	637163,5 add	
992123)	3687081	9920637163,5 the Root true to	
+ 37	2976369	the tenth Figure, and only	
9921267)	71071281	too much by an Unit in the	
+ 7 &c.	69448869	eleventh.	
992127*)	1622412		
....	992127		
	630285		
	595276		
	35009		
	29763		
	5246		
	4960		
	&c.		

* Here the Additions of the Quotient Figure being of no Consequence, therefore the Division is carried on from hence, as in page 68.

In the same manner the Cube Roots of Decimal Parts; or of Vulgar Fractions, being first changed into Decimals, may be extracted.

Sect. 4. To extract the Biquadrat Root.

IN extracting the Biquadrat Root, or that of the Fourth Power; (and indeed the Roots of all even Powers) there are some small Difficulties, not so easily expressed and explained in a few Words, as they are by an *Algebraick Theorem*; (such as shall be shewed further on) I have therefore in this Place made choice of extracting such Roots by two several Extractions, and the rather, because I presume the Reader by this Time thoroughly acquainted with the Business of extracting the Square Root, by which this may easily be performed. Thus:

First, Extract the Square Root of the proposed Resolvend, then the Square Root of that first Root will be the Biquadrat Root required.

Example 1. What is the Biquadrat Root of 4857532416? First extract it's Square Root,

Thus

Thus $\dot{4}8\dot{5}7\dot{5}3\dot{2}4\dot{1}6$

— $36 =$ the greatest Square, whose Root is 6.

1257532416 Remainder to be divided by 2.

First Root 6)	$\dot{6}2\dot{8}7\dot{6}6\dot{2}0\dot{8}$ (69696
$+\quad 9$	$\underline{5805}$
$\quad 69)$	$\underline{4826}$
$+\quad 6$	$\underline{4158}$
$\quad 696)$	$\underline{668620}$
$+\quad 9$	$\underline{626805}$
$\quad 6969$	$\underline{418158}$
	$\underline{418158}$
	(0)

Then $\dot{6}9\dot{6}9\dot{6}$ } being the first Root, whose Square Root
 — 4 } must now be extracted.

29696 Remainder to be divided by 2.

First Root 2)	14848 (264 the Biquadrat Root as was required.
$+\quad 6$	$\underline{138}$
$\quad 26)$	$\underline{1048}$
$+\quad 4$	$\underline{1048}$
$\quad 264$	$\underline{\quad\quad}$
	(0)

This is so easy I need not insert any more Examples.

Sect. 5. To extract the Sur-solid Root.

HAVING pointed the given Resolvend according as it's Index denotes; viz. into Periods of five Figures; seeking such a Sur-solid Number in the Table of Powers (or otherwise) as comes the nearest to the first Period of the Resolvend, whether greater or less, and call it's respective Root accordingly, viz. more than just, or less than just; annexing so many Cyphers to it, as there are remaining Periods of whole Numbers in the Resolvend; as before in extracting the Cube Root: Then find the Difference between the Resolvend, and the Sur-solid Number so taken, by subtracting the lesser from the greater (as before in the Cube). Next find the Cube of the aforesaid Sur-solid Root with it's annexed Cyphers, (which you may also do by the Table of Powers) and multiply that Cube with 5 the Index of the Sur-solid, the Product must be a Divisor, by which the Difference between the Resolvend and the Sur-solid Number must be divided; that so it may be depressed to

to a Square (as before in the Cube) which must be pointed into Periods of two Figures each, calling it the new Resolvend (as before). Then make the first Root, without it's Cyphers, a Divisor, enquiring how oft it may be found in the first Period of the new Resolvend, with this Consideration, if the Root (now a Divisor) be less than just, you must annex twice the Quotient Figure to it; but if it be more than just, you must subtract twice the Quotient Figure from a Cypher either annexed, or supposed to be annexed to that Divisor or Root, multiplying it so increased or diminished, with the said Quotient Figure, setting down their Product, &c. as before. An *Example* in each *Case* will render it plain and easy.

Example 1. Suppose it be required to extract the Surfolid Root out of this Number 12309502009375.

12309502009375 the Resolvend pointed.

The nearest Surfolid Number to 1230, the first Period of the Resolvend, is 1024, whose Root is 4 being less than just.

Therefore 12309502009375
 — 1024
 2069502009375 their Difference.

Next the Cube of 400 is 64000000 per Table, &c. And $64000000 \times 5 = 320000000$ the Divisor.

Then 320000000) 2069502009375 (6497 &c.

First Root	400	
+ 2 × 10 = +	20	
1 Divisor	420)	6467 (+ 400
+ 20 + 2 × 5 = +	30	15
	450	415 Root true
		2267
		2250

17 the Remainder to be rejected.

That is 415 is the Surfolid Root of the given Resolvend. As may be easily tried by involving it to the fifth Power. *Viz.* $415 \times 415 \times 415 \times 415 = 12309502009375$ the given Resolvend.

Note, Here again the double Quotient Figure ought to be twice added or subtracted, in the same Manner as the single one was directed for the Cube Root, page 131, and the Operation for the Surfolid Root in these two Examples is performed accordingly: contrary to what was heretofore done by the Author.

Example

Example 2. What is the Surfolid Root of 2327834559873

The nearest Surfolid Number to 232 is 243 whose Root is 3 being more than just.

$$\begin{array}{r} \text{Therefore } 2430000000000 \\ - 2327834559873 \\ \hline \end{array}$$

Remains 102165440127 For a Dividend.

The Cube of 300 is 27000000 and $27000000 \times 5 = 135000000$
Then 135000000) 102165440127 (756,7810 new Resolvend.

First Root	300		
$- 2 \times 2 =$	$- 4$		
1 Divisor	296)	756,7810	(300 -2,566
$- 4 - 2 \times 0,5 =$	$- 5,0$	592	297,434, The Root
2 Divisor	291,0)	164,78	only too little by 2
$- 1 - 2 \times 0,06 =$	$- 1,12$	145,50	in the lowest Fi-
3 Divisor	289,88)	19,2810	gure.
	&c.	&c.	

Now the Reason why this Root comes out to so many Places of Figures at the first Operation; is because the first Surfolid Number was so near the Resolvend, &c. As before.

Sect. 6. To extract the Root of the Square cubed.

THIS may be easily performed by two Extractions, according as it's Name denotes. Thus, first extract the Square Root of the given Resolvend; then extract the Cube Root of that Square Root, and it will be the Root required: That is, it will be the Root of the sixth Power. Or thus, first extract the Cube Root of the Resolvend; then extract the Cube Root of that Cube Root, and it will be the Root required.

E X A M P L E I.

Let it be required to extract the Square cubed Root out of this Number 145220537353515625 the Resolvend.

First I extract the Square Root of this Resolvend, which I take to be the best and easiest Way.

T

Thus

$$\begin{array}{r} \text{Thus} \quad 145220537353515625 \\ - \quad 9 \\ \hline \text{Remains} \quad 55220537353515625 \text{ to be halfed.} \end{array}$$

$$\begin{array}{r} \text{Then} \quad 3) \quad 27610268676757812,5 \quad (381078125 \\ + \quad 8 \quad 272 \\ \hline \quad 38) \quad 4102 \\ + \quad 10 \quad 3805 \\ \hline \quad 3810) \quad 2976867 \\ + \quad 7 \quad 2667245 \\ \hline \quad 38107) \quad 3096226 \\ + \quad 8 \quad 3048592 \\ \hline \quad 381078) \quad 47634757 \\ + \quad 1 \quad 38107805 \\ \hline \quad 3810781) \quad 95269528 \\ + \quad 2 \quad 76215622 \\ \hline \quad 38107812) \quad 1905390612,5 \\ + \quad 5 \quad 1905390612,5 \\ \hline \quad 381078125 \quad (0) \end{array}$$

Having found the Square Root of the given Resolvend, I proceed to extract the Cube Root of that Square Root.

$$\begin{array}{r} \text{That is, of} \quad 381078125 \\ - \quad 343 = \text{the nearest Cube, it's Root is } 700 \\ \hline \text{Then } 700 \times 3 = 2100) \quad 38078125 \quad (18161 \end{array}$$

First Root 7..

$$\begin{array}{r} + \quad 2 \\ 1 \text{ Divisor} \quad 72.) \\ + \quad 25 \\ 2 \text{ Divisor} \quad 745) \end{array} \quad \begin{array}{r} \quad \quad \quad (700 \\ 18161 \quad + \quad 25 \\ \hline 144 \quad 725 \\ 3761 \\ 3725 \\ \hline (36) \end{array}$$

Hence I find 725 to be the Square cubed Root required; as may easily be tried by involving it to the sixth Power. That is, $725 \times 725 \times 725 \times 725 \times 725 \times 725$ will be found = 14522053-7353515725 the given Resolvend.

Sect.

Sect. 7. To extract the Root of the seventh Power.

HAVING pointed the given Resolvend, as it's Index denotes, viz. into Periods of seven Figures, seek out such a Number of the seventh Power, by the Table of Powers, as comes nearest to the first Period of the Resolvend; whether it be greater or lesser, calling it's respective Root more than just, or less than just, annexing it's proper Number of Cyphers, &c. as in the Cube and Surfolid.

Then find the Difference between the given Resolvend, and that Number of the seventh Power (found by the Table of Powers) by subtracting the lesser from the greater.

Next find the Surfolid or fifth Power of that Root with it's annexed Cyphers (which you may also do by the Table of Powers) and multiply that Surfolid Number with 7, the Index of the given Resolvend; that Product must be a Divisor, by which the foresaid Difference must be divided, that so it may be depressed to a Square, to be pointed, &c. as before in the Cube, &c. then make the first Root, without it's Cyphers, a Divisor; working with it and the new Resolvend (as before) only here you must increase, or diminish the Divisor with thrice the Quotient Figure*.

Example. What is the second Surfolid Root, or that of the seventh Power,

of 382986553955078125 the Resolvend pointed.

— 2187 the nearest of the seventh Power.

164286553955078125 their Difference.

The first Root is 300 being less than just, and the fifth Power of 300 is 2430000000000, which being multiplied with 7 is 17010000000000 for a Divisor, by which the aforesaid Difference must be divided; which contracted may stand thus,

1701) 16428655 (9658,23 &c.

First Root	300	
+ 3 × 20 = +	60	
1 Divisor	360)	9658 (300
60 + 3 × 05 = +	75	+ 25
2 Divisor	435)	72
		325 = the true Root re-
		2458
		2175
		283 the Remainder to be rejected,
		[as before.]

* That is, by twice adding or subtracting the triple Quotient Figure, as was done with the double Quotient Figure for the Root of the fifth Power, page 136; and the single Quotient Figure for the Cube Root page 131.

Hence I have found 325 to be the true Root required, that is, the true Root of the seventh Power.

I think it needless to proceed farther; *viz.* to insert *Examples* of higher Powers. For if what is already done be well understood, it will be easy to conceive how to proceed in extracting the Root of any single Power how high soever it be (for the Method is general and alike in all Powers), due Regard being had to their Indices; and to the first single Side or Root. That is, whether it be more, or less than just, &c.

Yet methinks I hear the young Learner say, it is possible to follow the Directions and Examples, as they are here laid down; but still here is not the Reason why they are so, and so, performed; and why there should be a Remainder left after the Root is found; *viz.* when the given Resolvend hath a true Root of it's Kind.

It is true, the Reasons of these are not here laid down; neither indeed can they be rendered so plain and intelligible by Words, as by an Algebraick Process, from whence the *Theorems* or *Rules* here given, had their first Invention; as shall be shewed in the next Part, when I come to treat of resolving compounded or adaffected Equations; however, take this short and general Account of this Method.

This, and all other of the new Methods of Converging Series (as they are called), are very different from the former (and still common) Methods of extracting Roots, which require the first single Side or Root of the first Period (in any Resolvend) to be taken exactly true, and then by involving, and other tedious Ways of ordering it, there is formed a Divisor; which helps to grope out by Trials a second Figure in the Root. And so proceed on from Point to Point; still repeating the whole Work for every single Figure that comes into the Root. And if by Chance there be a Mistake or Error committed in any one Figure (as it is possible there may) it spoils the whole Process, which must then be wholly begun anew, or at least from that Part of it where the Error first entered.

But the Nature and Design of the Method which I have here laid down is quite otherwise; it being so contrived, as to gradually lessen the Difference betwixt any proposed Power, and the like Power of another Number assumed; *viz.* it lessens that Difference until it is either quite vanquished, or becomes so infinitely small as to be insignificant.

Therefore when any Number is proposed to have it's Root extracted; it is here required to take the next nearest Root of the first Period in the Resolvend; that so the Difference betwixt the given

given Resolvend, and the Homogeneous Power (*viz.* the like Power) of the Root thus taken, may be less either in Excess, or Defect. Which Difference being reduced, or depressed lower, becomes so prepared, that by plain Division (comparatively) there will arise such Quotient Figures, as will both correct and increase the first Root to three Places of Figures at least, sometimes to four or five Places of Figures; according as the said first Difference happens to be more or less (of which you may have observed Instances): But yet there will be a Remainder left, and perhaps an Excess or Defect in the Root so increased, *viz.* in the last Figure of it.

Now to rectify the said Excess or Defect in the Root, and to discover whether the given Resolvend be a true Figurate Number, or not: That is, whether it have a true Root of it's kind; it will be necessary to make a second Operation; by taking the Root so increased, and proceeding with it and the given Resolvend, in all respects as in the first Work (like to the third *Example* of extracting the Cube Root): I say, if the given Resolvend have a true Root, it will appear at this second Operation, and all the aforesaid Differences, &c. will be vanquished; provided the Root required is not to have more than three (or four) Places of Figures in it.

But if the Root be to have more than three Figures in it; or, that the given Resolvend prove to be a Surd Number. Then there will be a Difference as before; which will afford Quotient Figures to rectify and increase the Root last taken, to three Times as many Places of Figures, as it had at the Beginning of that second Operation. As you may see in the aforesaid *Example 3.* of the Cube Root; wherein that Root is increased to twelve Places of Figures at two Operations; which if it were to be extracted the old (and still common) way, it would require at least forty times the Number of Figures I have here used.

Again, if there chance to be a Mistake committed in any Operation performed by the Method here laid down, that Mistake will not destroy the precedent Work, but will be rectified in the next Operation, although it were not discovered before. And thus you may proceed on to a third Operation, which will afford 27 Places of Figures in the Root, &c. with very little Trouble, if compared with former Methods.

The brief Account, which I have here given (*by Way of explaining the Nature of this Method of extracting Roots*) being well considered and compared with the several Operations of the foregoing *Examples*, must needs help the Learner to form such an Idea of it, that he cannot (I presume) but understand how to proceed

proceed in extracting the Root out of any single Power, how high soever it be; without the Help of an *Algebraick Theorem*. Not but when that comes to be once understood; the Work will be much readier and easier performed: As will appear in the next Part.

I did intend to have here inserted the whole Business of Interest and Annuities; but finding that it would require too large a Discourse, to shew the Grounds and Reasons of the several *Theorems* useful therein, I have therefore reserved that Work for the Close of the next Part. Neither indeed can the raising of those *Theorems* be so well delivered in Words, as by an *Algebraick Way* of arguing; which renders them not only much shorter, but also plainer and easier to be understood.

I have also omitted that Rule in *Arithmetick*, usually called the *Rule of Position*, or *Rule of False*: Because all such Questions, as can be answered by that guessing Rule, are much better done by any one who hath but a very small smattering of *Algebra*. I shall therefore conclude this Part of *Numerical Arithmetick*; and proceed to that of *Algebraick Arithmetick*, wherein I would advise the young Learner not to be too hasty in passing from one Rule to another, and then he will find it very easy to be attained.

A N

INTRODUCTION

T O T H E

Mathematicks.

P A R T II.

P R O Æ M.

HAVING formerly wrote a small *Traët* of *Algebra*, perhaps it may seem (to some) very improper to write again upon the same Subject; but only (as the usual Custom is) to have referred my Reader to that *Traët*. However, because the following Parts of this Treatise are managed by an *Algebraick* Method of arguing; which may fall into the Hands of those who have not seen that *Traët*, or any other of that Kind; I thought it convenient to accommodate the young Geometer with the first Elements, or principal Rules, by which all Operations in this Art are performed; that so he may not be at a Loss as he proceeds farther on: Besides, what I formerly wrote was only a Compendium of that which is here fully handled at large.

The principal Rules are *Addition*, *Subtraction*, *Multiplication*, *Division*, *Involution*, and *Evolution*, as in common *Arithmetick*, but differently performed; and therefore some call it *Algebraick Arithmetick*. Others call it *Arithmetick in Specie*, because all the Quantities concerned in any Question, remain in their substituted Letters (howsoever managed by *Addition*, *Subtraction*, or *Multiplication*, &c.) without being destroyed or changed into others, as Figures in common *Arithmetick* are.

Mr. *Harriot* called it *Logistica Speciosa*, or *Specious Computation*.

C H A P.

C H A P. I.

*Concerning the Method of noting down Quantities ;
and tracing their Steps, &c.*

Sect. 1. Of Notation.

THE Method of noting down Letters for Quantities is various, according to every one's Fancy ; but I shall here follow the same as in my former Tract, and represent the Quantity sought (be it Line or Number, &c.) by the small (*a*), and if more Quantities than one are sought, by the other small Vowels, *e. u. or y.*

The given Quantities are represented by the small Consonants, *b. c. d. f. g. &c.*

And for Distinction sake, mark the Points or Ends of Lines in all Schemes, with the capital or great Letters, viz. *A. B. C. D. &c.*

When any Quantity (either given or sought) is taken more than once, you must prefix it's Number to it ; as $3a$ stands for *a* taken three times, or three times *a*, and $7b$ stands for seven times *b*, &c.

All Numbers thus prefixt to any Quantity, are called Coefficients or Fellow-Factors ; because they multiply the Quantity ; and if any Quantity be without a Coefficient, it is always supposed or understood to have an Unit prefixed to it ; as *a* is $1a$, or *b* is $1b$, &c.

The Signs by which Quantities are chiefly managed are the same, and have the same Signification, with those in the first Part, page 5. which I here presume the Reader to be very well acquainted with. To them must be here added these three more ;

Viz. $\left\{ \begin{array}{l} \textcircled{\text{C}} \\ \omega \\ \sqrt{} \end{array} \right\}$ the Sign of $\left\{ \begin{array}{l} \text{Involution.} \\ \text{Evolution, or extracting Roots.} \\ \text{Irrationality, or Sign of a Surd Root.} \end{array} \right.$

All Quantities that are expressed by Numbers only (as in *Vulgar Arithmetick*) are called *Absolute Numbers*.

Those Quantities that are represented by single Letters, as, *a. b. c. d. &c.* or by several Letters that are immediately joined together ; as *ab. cd. or 7bd. &c.* are called Simple or Single whole Quantities.

But when different Quantities represented by different or unlike Letters, are connected together by the Signs (+ or —) ; as $a + b$, $a - b$, or $ab - dc$, &c. they are called Compound whole Quantities.

And

And when Quantities are expressed or set down like Vulgar Fractions, Thus $\frac{a}{b}$, or $\frac{a+b}{d}$, or $\frac{ab+dc}{b-c}$, &c. they are called Fractional or Broken Quantities.

The Sign wherewith Quantities are connected, always belongs to that Quantity which immediately follows it; and therefore all the Quantities concerned in any Question, may stand in any order at Pleasure, viz. the most convenient for the next Operation. As $a+b-d$ may stand thus $b-d+a$, or thus $a-d+b$, or $-d+a+b$ &c. these being still the same, tho' differently placed.

That Quantity which hath no Sign before it (as generally the leading Quantity hath not) is always understood to have the Sign $+$ before it. As a is $+a$, or $b-d$ is $+b-d$, &c. for the Sign $+$ is the Affirmative Sign, and therefore all leading or positive Quantities are understood to have it, as well as those that are to be added.

But the Sign $-$ being the Negative Sign, or Sign of Defect, there is a Necessity of prefixing it before that Quantity to which it belongs, wherever the Quantity stands.

Sect. 2. Of tracing the Steps used in bringing Quantities to an Equation.

THE Method of tracing the Steps, used in bringing the Quantities concerned in any Question to an Equation, is best performed by registering the several Operations with Figures and Signs placed in the Margin of the Work, according as the several Operations require; being very useful in long and tedious Operations.

For Instance: If it be required to set down and register the Sum of the two Quantities, a and b , the Work will stand,

Thus $\begin{array}{r|l} 1 & a \\ 2 & b \\ \hline 1+2 & 3 \end{array}$ First set down the proposed Quantities, a and b , over-against the Figures 1, 2, in the small Column (which are here called Steps), and against 3 (the third Step), set down their Sum, viz. $a+b$. Then against that third Step, set down $1+2$ in the Margin; which denotes that the Quantities against the first and second Steps are added together, and that those in the third Step are their Sum.

To illustrate this in Numbers, suppose $a=9$, and $b=6$. Then it will be,

Thus $\begin{array}{r|l} 1 & a = 9 \\ 2 & b = 6 \\ \hline 1+2 & 3 \end{array}$ $a+b=9+6=15$ being the Sum of 9 and 6.

U

Again,

C H A P. I.

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Viz. $\left\{ \begin{array}{c} \odot \\ \omega \\ \checkmark \end{array} \right\}$ the Sign of $\left\{ \begin{array}{l} \text{Involution.} \\ \text{Evolution, or extracting Roots.} \\ \text{Irrationality, or Sign of a Surd Root.} \end{array} \right.$

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For Instance: If it be required to set down and register the Sum of the two Quantities, a and b , the Work will stand,

Thus $\begin{array}{l|l} 1 & a \\ 2 & b \\ \hline 1+2 & 3 \end{array}$ First set down the proposed Quantities, a and b , over-against the Figures 1, 2, in the small Column (which are here called Steps), and against 3 (the third Step), set down their Sum, *viz.* $a+b$. Then against that third Step, set down $1+2$ in the Margin; which denotes that the Quantities against the first and second Steps are added together, and that those in the third Step are their Sum.

To illustrate this in Numbers, suppose $a=9$, and $b=6$. Then it will be,

Thus $\begin{array}{l|l} 1 & a = 9 \\ 2 & b = 6 \\ \hline 1+2 & 3 \end{array}$ $a+b=9+6=15$ being the Sum of 9 and 6.

U

Again,

Again, If it were required to set down the Difference of the same two Quantities ; then it will be,

$$\begin{array}{r|l} \text{Thus} & \begin{array}{l} 1 \quad a = 9 \\ 2 \quad b = 6 \\ \hline 1 - 2 \quad 3 \quad a - b = 9 - 6 = 3 \end{array} \end{array} \text{ the Diff. between 9 and 6.}$$

Or if it were required to set down their Product.
Then it will be,

$$\begin{array}{r|l} \text{Thus} & \begin{array}{l} 1 \quad a = 9 \\ 2 \quad b = 6 \\ \hline 1 \times 2 \quad 3 \quad a \times b \text{ or } ab = 9 \times 6 = 54 \end{array} \end{array} \text{ the Prod. of 9 into 6.}$$

&c.

Note, Letters set or joined immediately together (like a Word) signify the Rectangle or Product of those Quantities they represent ; as in the last Example, wherein $ab = 54$ is the Product of $a = 9$ and $b = 6$. &c.

Axioms.

1. If equal Quantities be added to equal Quantities, the Sum of these Quantities will be equal.
2. If equal Quantities be taken from equal Quantities, the Quantities remaining will be equal.
3. If equal Quantities be multiplied with equal Quantities, their Products will be equal.
4. If equal Quantities be divided by equal Quantities, their Quotients will be equal.
5. Those Quantities, that are equal to one and the same Thing, are equal to one another.

Note, I advise the Learner to get these five Axioms perfectly by Heart.

These Things being premised, and a perfect Knowledge of the Signs and their Significations being gained, the young *Algebraist* may proceed to the following *Rules*. But first I must make bold to advise him here (as I have formerly done) that he be very ready in one Rule before he undertakes the next.

That is, He should be expert in *Addition*, before he meddles with *Subtraction* ; and in *Subtraction*, before he undertakes *Multiplication*, &c. because they have a Dependency one upon another.

C H A P. II.

Concerning the Six Principal Rules, of Algebraick Arithmetick, of whole Quantities.

Sect. I. Addition of whole Quantities.

ADDITION of whole Quantities admits of three Cases.

Case 1. If the Quantities are like, and have like Signs; add the Co-efficients or prefix Numbers together, and to their Sum adjoin the Quantities with the same Sign.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
	1	a	$-a$	$5b$	$-7bc$
	2	a	$-a$	$3b$	$-8bc$
1 + 2	3	$2a$	$-2a$	$8b$	$-15bc$

Thus		Exam. 5.	Exam. 6.	Exam. 7.
	1	$3a + 5b$	$3a - 5b$	$6ab + 12$
	2	$2a + 7b$	$2a - 7b$	$3ab + 24$
1 + 2	3	$5a + 12b$	$5a - 12b$	$9ab + 36$

The Reason of these Additions is evident from the Work of Common Arithmetick. For suppose a , to represent one Crown, to which if I add one Crown, the Sum will be two Crowns, or $2a$. as in Example 1.

Or if we suppose $-a$, to represent the Want or Debt of one Crown, to which if another Want or Debt of one Crown be added, the Sum must needs be the Want or Debt of two Crowns; or $-2a$; as in Example 2. And so for all the rest.

Case 2. If the Quantities are alike, and have unlike Signs; subtract the Co-efficients, from each other, and to their Difference join the Quantities with the Sign of the greater.

		Exam. 8.	Exam. 9.	Exam. 10.	Exam. 11.
	1	$+5a$	$-5a$	$7bc$	$-9abd$
	2	$-2a$	$+3a$	$-6bc$	$+7abd$
1 + 2	3	$+2a$	$-2a$	$-1bc$	$-2abd$

		Example 12.	Example 13.
	1	$7a - 5b$	$-8ab - 7bc + 15$
	2	$-5a + 7b$	$+12ab + 7bc - 24$
1 + 2	3	$2a + 2b$	$4ab - 9$

The Reason of the Operations in this Case may be easily understood, by any one that duly considers the comparing of Stock and Debts together, or the ballancing of Accounts betwixt Debtor and Creditor. That is, the Affirmative Quantities represent the Stock or Creditor: The Negative Quantities represent the Debts; and their Sum represents the Ballance, &c.

Case 3. When the Quantities are unlike, set them all down, without altering their Signs; and thence will arise compound Quantities, which can be no otherwise added but by their Signs.

$$\begin{array}{r|l|l|l} \text{Thus} & 1 & a & a \\ & 2 & b & -b \\ \hline 1+2 & 3 & a+b & a-b \\ & & & 5b+7dc \\ & & & 4a+20 \end{array}$$

Here follow a few Examples wherein all the 3 Cases are promiscuously concerned.

$$\begin{array}{r|l|l|l} & 1 & aa+2ab+bb & 8ab+bc-37 \\ & 2 & -4ab & -7ab-bc+42-6d \\ \hline 1+2 & 3 & aa-2ab+bb & ab+5-6d \end{array}$$

$$\begin{array}{r|l|l|l} & 1 & aa-2ab+bb & 9bc+7ab-45 \\ & 2 & +4ab+bb & 4d-6bc-7ab+da \\ \hline 1+2 & 3 & aa+2ab+bb & 3bc+4d-45+da \end{array}$$

$$\begin{array}{r|l|l|l} & 1 & 5a & a+b-ab \\ & 2 & -7a & 7c-d \\ & 3 & +3a & 4e+f \\ \hline 1+2+3 & 4 & a & a+b-ab+7c-d+4e+f \end{array}$$

$$\begin{array}{r|l|l|l} & 1 & 3aa+4abc-bb+30 \\ & 2 & 2bb-3aa-2abc-25 \\ & 3 & dd+2aa-3abc-3 \\ \hline 1+2+3 & 4 & +dd+2aa+bb-abc+2 \end{array}$$

Sect. 2. Subtraction of whole Quantities.

SUBTRACTION of whole Quantities is performed by one general Rule.

R U L E.

Change all the Signs of the Subtrahend (viz. of those Quantities which are to be subtracted) or suppose them in your Mind to be changed; then add all the Quantities together, as before in Addition, and their Sum will be the true Remainder, or Difference required.

This

This general Rule is deduced from these evident Truths.

To subtract an Affirmative Quantity, from an Affirmative, is the same as to add a Negative Quantity to an Affirmative: that is $+2a$ taken from $+3a$, is the same with $-2a$ added to $+3a$. Consequently, to subtract a Negative Quantity from an Affirmative, will be the same as to add an Affirmative Quantity to an Affirmative: that is $-2a$ taken from $+3a$ will be the same with $+2a$ added to $+3a$.

		Exam. 1.	Exam. 2.	Exam. 3.	Exam. 4.
1 — 2	1	$2a$	$-2a$	$8b$	$-15bc$
	2	a	$-a$	$3b$	$-8bc$
	3	a	$-a$	$5b$	$-7bc$

		Exam. 5.	Exam. 6.	Exam. 7.
1 — 2	1	$5a + 12b$	$5a - 12b$	$9ab + 36$
	2	$2a + 7b$	$2a - 7b$	$3ab + 24$
	3	$3a + 5b$	$3a - 5b$	$6ab + 12$

		Exam. 8.	Exam. 9.	Exam. 10.	Exam. 11.
1 — 2	1	$+2a$	$-2a$	bc	$-2abd$
	2	$-3a$	$+3a$	$-6bc$	$+7abd$
	3	$+5a$	$-5a$	$+7bc$	$-9abd$

		Example 12.	Example 13.
1 — 2	1	$2a + 2b$	$4ab - 9$
	2	$-5a + 7b$	$-8ab - 7bc + 15$
	3	$7a - 5b$	$12ab + 7bc - 24$

If these 13 Examples be compared with those in Addition, the Work will appear very evident, these being only the Converse or Proof of those; according to the Nature of Addition and Subtraction in common Arithmetick.

More Examples in Subtraction.

1 — 2	1	$a + b$	$5bc + 3da$	$8a + 5bd + 25$
	2	$a - b$	$5bc - 4da$	$7a - 3bd - 12$
	3	$+2b$	$+7da$	$a + 8bd + 37$

	1	$c + 13$	a	0
	2	$3a - b - 2c$	b	$2a - 4b$
1 - 2	3	$3c + 13 - 3a + b$	$a - b$	$-2a + 4b$
	2	$a + b - 54$		76
	1	$d - 3b - bc - 75$		$a - b - 5d + 7c$
1 - 1	3	$a + 4b + bc + 21 - d$	$76 - a + b + 5d - 7c$	

That $a - b$ taken from $a + b$ leaves $+ 2b$ for the Remainder, as in the first of these *Examples*, may be thus proved :

Let	1	$a + b = z$	
And	2	$a - b = x$	
$2 + b$	3	$a = x + b$	per Axiom 1.
$1 - 3$	4	$b = z - x - b$	per Axiom 2.
$4 + b$	5	$2b = z - x$	which was to be proved.

The Truth of all Operations in *Subtraction*, where any Doubt arises, may be proved, by adding the Subtrahend to the Remainder, as in Common *Arithmetick*.

E X A M P L E.

From	1	$+ 5a$	0	$- 9bc$	
Take	2	$- 2a$	$+ 3b$	$- 6da$	Subtrahend.
1 - 2	3	$+ 7a$	$- 3b$	$+ 6da - 9bc$	Remainder.
$2 + 3$	4	$+ 5a$	0	$9bc$	Proof.

Sect. 3. Multiplication of whole Quantities.

MULTIPLICATION of whole Quantities admits of three Cases.

Case 1. When the Quantities have like Signs, and no Coefficients, set or join them together, and prefix the Sign $+$ before them; and that will be their Product.

		<i>Exam. 1.</i>	<i>Exam. 2.</i>	<i>Exam. 3.</i>	<i>Exam. 4.</i>
Thus	1	a	$- a$	$a + b$	$- a - b$
	2	b	$- b$	d	$- d$
1×2	3	ab	$+ ab$	$ad + bd$	$+ ad + bd$

Case 2. If there be Coefficients; multiply them, and to their Product adjoin the Quantities set together as before.

Thus

		Exam. 5.	Exam. 6.	Exam. 7.	Exam. 8.
Thus {	1	$5a$	$-6d$	$3a + 2b$	$a + b$
	2	$3b$	$-7b$	6	$5b$
	1×2	$15ab$	$+42db$	$18a + 12b$	$5ab + 5bb$

Case 3. When the Quantities have unlike Signs; join them and the Product of the Coefficients together (as before), but prefix the Sign — before them;

		Exam. 9.	Exam. 10.	Exam. 11.	Exam. 12.
Thus {	1	$+a$	$-6d$	$4a - 7b$	$4a - 7b$
	2	$-b$	$+7b$	$3f$	$-3f$
	1×2	$-ab$	$-42db$	$12af - 21bf$	$-12af + bf$

That is, $+$ into $+$, or $-$ into $-$, gives $+$
But $+$ into $-$, or $-$ into $+$, gives $-$ } in the Product.

That $+$ into $+$ will produce $+$ in the Product, is evident from *Multiplication in Common Arithmetick*: viz. $+5$ into $+7$ will give $+35$ &c. But that $+$ into $-$, or $-$ into $+$ should produce the Sign $-$, as in the four last Examples; and that $-$ into $-$ should produce the Sign $+$, as in the second, fourth, and sixth Examples, may perhaps seem somewhat hard to be conceived; and requires a Demonstration.

First to prove that $-7b$ into $+3f = -21bf$. As in Example 11.

Suppose	1	$4a - 7b = 0$	
Then will	2	$4a = 7b$	per Axiom 1.
But	3	$+3f = +3f$	
2×3	4	$12af = 21bf$	per Axiom 3.
$4 - 21bf$	5	$12af - 21bf = 0$	per Axiom 2.

Consequently $+$ into $-$, or $-$ into $+$ produces $-$, which was the Thing to be proved.

Secondly to prove that $-7b$ into $-3f$ gives $+21bf$ as in Example 12.

Let	1	$4a - 7b = 0$	} as before.
Then	2	$4a = 7b$	
But	3	$-3f = -3f$	
the 2×3 is	4	$-12af = -21bf$	by what is proved above.
$4 + 21bf$	5	$-12af + 21bf = 0$	per Axiom 1.

Consequently $-$ into $-$ gives $+$, which was to be proved.

Or

Or these may be otherwise proved by Numbers.

Thus, suppose $\begin{cases} a = 20 \\ b = 14 \end{cases}$ and $\begin{cases} c = 12 \\ d = 8 \end{cases}$ or any other Numbers.

Then $\overline{a - b = 6}$ & $\overline{c - d = 4}$ per Axiom 2.

Consequently, $\overline{a - b} \times \overline{c - d} = 6 \times 4 = 24$, per Axiom 3.
but $\overline{a - b} \times \overline{c - d}$, according to the precedent Rules, will be,
 $\overline{ac - cb + bd - da}$, which, if true, must be equal to 24.

Proof $\begin{cases} ac = 20 \times 12 = 240 & cb = 12 \times 14 = 168 \\ bd = 14 \times 8 = 112 & da = 8 \times 20 = 160 \end{cases}$

Hence $\overline{ac + bd = 352}$ per Axiom 1.

And $\overline{cb + da = 328}$ which being subtracted,

Leaves $\overline{ac + bd - cb - da = 352 - 328 = 24}$, which plainly shews,

That $+$ into $-$ produces $-$ } in the Product.
And $-$ into $-$ produces $+$ }
Q. E. D.

Note, If the Multiplier consists of several Terms, then every one of those Terms must be multiplied into all the Terms of the Multiplicand; and the Sum of those particular Products, will be the Product required, as in Common *Arithmetick*.

EXAMPLES.

I × a	1	$a + b - d$	$7b + 5d$
	2	$a - b$	$3a - 5f$
	3	$aa + ba - da$	$21ba + 15da$
	4	$-ba - bb + db$	$-35bf - 25df$
$3 + 4$	5	$aa - da - bb + db$	$21ba + 15da - 35bf - 25df$

I × 2	1	$aa - ba$	$2c - 2d$
	2	$a + b$	$3a - 4b$
	3	$aaa - abb$	$6ca - 9da - 8bc + 12db$

I × 2	1	$aa + 2a + 4$	$aa - ba + bb$
	2	$a - 2$	$a + b$
	3	$aaa + 2aa + 4a$	$aaa - baa + bba$
	4	$-2aa - 4a - 8$	$+baa - bba + bbb$
3	3	$aaa - 8$	$aaa + bbb$

Sect.

Sect. 4. Division of whole Quantities.

Division of Species, is the converse or direct contrary to that of *Multiplication*, and consequently is performed by converse Operations (as in common *Arithmetick*), and admits of four *Cases*.

Case 1. When the Quantities in the Dividend, have like Signs to those in the Divisor, and no Co-ëfficients in either; cast off or expunge all the Quantities in the Dividend, that are like those in the Divisor; and set down the other Quantities with the Sign + for the Quotient required.

$$\begin{array}{l} \text{Thus } \left\{ \begin{array}{l|l|l|l|l} 1 & ab & -ab & ad+bd & -ad-bd \\ 2 & b & -b & d & -d \\ \hline 1 \div 2 & 3 & a & a+b & a+b \end{array} \right. \end{array}$$

Case 2. When the Quantities in the Dividend have unlike Signs to those in the Divisor; then set down the Quotient Quantities found as before, with the Sign — before them.

$$\begin{array}{l} \text{Thus } \left\{ \begin{array}{l|l|l|l} 1 & +ab & -ab-bd & abc+bcd+bcf \\ 2 & -b & +b & -bc \\ \hline 1 \div 2 & 3 & -a & -a-d-f \end{array} \right. \end{array}$$

Case 3. If the Quantities in the Dividend and Divisor, have Co-ëfficients; divide the Numbers (as in common *Arithmetick*) and to their Quotients adjoin the Quotient Quantities.

$$\begin{array}{l} \text{Thus } \left\{ \begin{array}{l|l|l|l} 1 & 15ab & 42db & 12af-21bf \\ 2 & 3b & -7b & 3f \\ \hline 1 \div 2 & 3 & 5a & -6d & 4a-7b \end{array} \right. \end{array}$$

Note, When the Quantities and Co-ëfficients in the Divisor and Dividend are all the same, the Quotient will be an Unit, or 1.

$$\begin{array}{l} \text{Thus } \left\{ \begin{array}{l|l|l|l} 1 & ab & 9bc & 7ab+5bc & 8ab+4d \\ 2 & ab & -9bc & 7ab+5bc & -8ab-4d \\ \hline 1 \div 2 & 3 & 1 & -1 & 1 & -1 \end{array} \right. \end{array}$$

Case 4. When the Quantities in the Divisor cannot be exactly found in the Dividend; then set them both down like a *Vulgar Fraction*, as in common *Arithmetick*.

X

Thus

$$\begin{array}{l} \text{Thus } \left\{ \begin{array}{l} 1 \mid a \mid 6bc \mid 5b + aa \mid 8adc \\ 2 \mid b \mid 3d \mid 5d + 7b \mid 4abc \end{array} \right. \\ 1 \div 2 \left\{ \begin{array}{l} 3 \mid \frac{a}{b} \mid \frac{2bc}{d} \mid \frac{5b + aa}{5d + 7b} \mid \frac{2d}{b} \end{array} \right. \end{array}$$

N. B. In Division one thing must be very carefully observed; viz. that like Signs give + and unlike Signs give — in the Quotient; which needs no other Proof than that already laid down in the last Section, if duly compared with what hath been said concerning *Multiplication* and *Division*, in *Vulgar Arithmetick*.

Examples of Division at large.

$$\begin{array}{r|l} 1 & 21ba + 15da - 35bf - 25df (+ 3a \\ 2 & 7b + 5d \\ \hline 2 \times 3a & 3 \mid 21ba + 15da \\ 1 - 3 & 4 \mid 0 \quad 0 \quad - 35bf - 25df (- 5f \\ 2 \times -5f & 5 \mid \quad \quad - 35bf - 25df \\ 4 - 5 & 6 \mid \quad \quad 0 \quad 0 \\ 1 \div 2 & 7 \mid 3a - 5f \text{ the Quot. collected from the 3 and 5 Steps.} \end{array}$$

Or *Division* of Quantities may stand as Numbers in common *Arithmetick* do; thus

$$\begin{array}{r} 3a - 6 \mid 6aaaa - 96 \quad (2aaa + 4aa + 8a + 16 \\ \quad 6aaaa - 12aaa \\ \hline \quad 0 + \quad 12aaa - 96 \\ \quad + \quad 12aaa - 24aa \\ \hline \quad \quad 0 + 24aa - 96 \\ \quad \quad + 24aa - 48a \\ \hline \quad \quad \quad 0 + 48a - 96 \\ \quad \quad \quad + 48a - 96 \\ \hline \quad \quad \quad \quad 0 \quad 0 \end{array}$$

That is, $6aaaa - 96 \div 3a - 6$ gives $2aaa + 4aa + 8a + 16$ for the Quotient, as may easily be proved by *Multiplication*, viz. $2aaa + 4aa + 8a + 16 \times 3a - 6$ will produce $6a^4 - 96$; and so for the rest.

Sect. 5. *Involution of whole Quantities.*

INvolution is the raising or producing of Powers, from any proposed Root, and is performed in all respects like *Multiplication*, save only in this; *Multiplication* admits of any different Factors, but *Involution* still retains the same.

E X A M P L E S.

E X A M P L E S.

	1	a	$-a$	the Root, or single Power.
I $\odot^2$	2	aa	$+aa$	Square, or second Power.
I $\odot^3$	3	aaa	$-aaa$	Cube, or third Power.
I $\odot^4$	4	$aaaa$	$+aaaa$	Biquadrat, or fourth Power.
I $\odot^5$	5	$aaaaa$	$-aaaaa$	Surfolid, or fifth Power, &c.

Note, The Figures placed in the Margin, after the Sign ($\odot$) of Involution, shew to what Height the Root is involved; and are called Indices of the Power; and are usually placed over the involved Quantities, in order to contract the Work, especially when the Powers are any thing high.

$$\text{Thus } \begin{cases} a = a \\ a^2 = aa \\ a^3 = aaa \\ a^4 = aaaa \end{cases} \quad \text{And } \begin{cases} a^5 = aaaaa \\ a^6 = aaaaaa \\ a^5 b^5 = aaaaa bbbbbb \\ a^3 b^3 d^3 = aaa bbb ddd \end{cases}$$

If the Quantities have Co-efficients, the Co-efficients must be involved along with the Quantities, as in these,

Thus	1	$2a$	$-3a$	$5bc$
I $\odot^2$	2	$4aa$	$+9aa$	$25bbcc$
I $\odot^3$	3	$8aaa$	$-27aaa$	$125bbbccc$
I $\odot^4$	4	$16aaaa$	$+81aaaa$	$625b^4c^4$
I $\odot^5$	5	$32aaaaa$	$-243a^5$	$3125b^5c^5$ &c.

Involution of Compound Quantities is performed in the same manner, due regard being had to their Signs and Co-efficients, if there be any. As for instance, suppose $a + b$ were given to be involved to the fifth Power.

Thus	1	$a + b$ called a Binomial Root
		$a + b$
I $\times a$	2	$aa + ab$
I $\times b$	3	$+ab + bb$
I $\odot^2$	4	$aa + 2ab + bb$, the Square of $a + b$
		$a + b$
4 $\times a$	5	$aaa + 2aab + abb$
4 $\times b$	6	$+ aab + 2abb + bbb$
I $\odot^3$	7	$aaa + 3aab + 3abb + bbb$, the Cube of $a + b$
		X 2
		aaa

	7	$aaa + 3aab + 3abb + bbb$ $a + b$
$7 \times a$	8	$a^4 + 3a^3b + 3aabb + abbb$
$7 \times b$	9	$+ a^3b + 3aabb + 3abbb + b^4$
$1 \odot^5$	10	$a^4 + 4a^3b + 6aabb + 4abbb + b^4$ $a + b$
$10 \times a$	11	$a^5 + 4a^4b + 6a^3bb + 4aabb^3 + ab^4$
$1 \times b$	12	$a^4b + 4a^3bb + 6aabb^3 + ab^4 + b^5$
$1 \odot^5$	13	$a^5 + 5a^4b + 10a^3bb + 10aabb^3 + 5ab^4 + b^5$ &c.

Again, Let $a - b$, called a Residual Root, be given.

Then	1	$a - b$ $a - b$
$1 \times a$	2	$aa - ab$
$1 \times -b$	3	$-ab + bb$
$1 \odot^2$	4	$aa - 2ab + bb$ the Square of $a - b$ $a - b$
$4 \times a$	5	$aaa - 2aab + abb$
$4 \times -b$	6	$-aab + 2abb - bbb$
$1 \odot^3$	7	$aaa - 3aab + 3abb - bbb$, the Cube of $a - b$ $a - b$
$7 \times a$	8	$aaaa - 3aaab + 3aabb - abbb$
$7 \times -b$	9	$-aaab + 3aabb - 3abbb + bbbb$
$1 \odot^4$	10	$aaaa - 4aaab + 6aabb - 4abbb + bbbb$ $a - b$
$10 \times a$	11	$a^5 - 4a^4b + 6a^3bb - 4aabb^3 + ab^4$
$10 \times -b$	12	$-a^4b + 4a^3bb - 6aabb^3 + 4ab^4 + b^5$
$1 \odot^5$	13	$a^5 - 5a^4b + 10a^3bb - 10aabb^3 + 5ab^4 - b^5$ &c.

By comparing these two Examples together, you may make the following Observations.

1. That the Powers raised from a Residual Root (*viz.* the Difference of two Quantities) are the same with their like Powers raised from a Binomial Root (or the Sum of two Quantities) save only in their Signs; *viz.* the Binomial Powers have the Sign + to every Term, but the Residual Powers have the Signs + and - interchangeably to every other Term.

2. The Indices of the Powers of the leading Quantity (a) continually decrease in Arithmetical Progression; *viz.* in the Square it

it is aa, a : In the Cube aaa, aa, a : In the Biquadrat $aaaa, aaa, aa, a$, &c.

3. The Indices of the other Quantity b do continually increase in Arithmetical Progression; viz. In the Square it is b, bb : In the Cube b, bb, bbb : In the Biquadrat $b, bb, bbb, bbbb$, &c.

4. The first and last Terms, are always pure Powers of the single Quantities, and are both of the same Height.

5. The Sum of the Indices of any two Letters joined together in the intermediate Terms, are always equal to the Index of the highest Power, viz. of the first or last Term.

These Observations being duly considered, it will be easy to conceive how the Terms of any proposed Power raised from a Binomial or Residual Root must stand, without their Unciæ or Numerical Figures.

For Instance, suppose it were required to raise the Binomial Root $a + b$ to the seventh Power; then the Terms of that Power will stand without their Unciæ in this Order.

$$\text{Viz. } a^7 + a^6 b + a^5 b^2 + a^4 b^3 + a^3 b^4 + a^2 b^5 + a b^6 + b^7.$$

And because the Uncia (not only of any single Letter, but also) of every single Power, how high soever it be, is an Unit or 1 (which neither multiplies nor divides) and all the Powers of any Binomial or Residual Root are naturally raised by multiplying of the precedent Power into it's original Root, which is done by only joining each Letter in the Root to the precedent Power, with it's Unciæ, and then removing the said Power, when it is so joined to the second Letter, one place forward (either to the left or right Hand) it must needs follow,

That the Unciæ of the second Terms (in any such Power) will always be the Sum of so many Units added together more one, as there have been Multiplications of the first Root; which will always be determined by the Index of the first Term in the Power.

And because the Unciæ of all the intermediate Terms, are only removed along with their Letters, it also follows; that if they are added together, their respective Sums will produce the true Unciæ of the intermediate Terms in the new raised Power. As doth plainly appear from the following Numbers so removed without their Letters; which both shews and demonstrates an easy Way of producing the Unciæ of any ordinary Power (viz. of one not very high) raised from either a Binomial or Residual Root.

Thus

Thus

Add $\left\{ \begin{array}{c} 1 \cdot 1 \cdot \\ \quad 1 \cdot 1 \end{array} \right.$ The two Unciæ of the Root.

Add $\left\{ \begin{array}{c} 1 \cdot 2 \cdot 1 \\ \quad 1 \cdot 2 \cdot 1 \end{array} \right.$ The Unciæ of the Square.

Add $\left\{ \begin{array}{c} 1 \cdot 3 \cdot 3 \cdot 1 \\ \quad 1 \cdot 3 \cdot 3 \cdot 1 \end{array} \right.$ The Unciæ of the Cube.

Add $\left\{ \begin{array}{c} 1 \cdot 4 \cdot 6 \cdot 4 \cdot 1 \\ \quad 1 \cdot 4 \cdot 6 \cdot 4 \cdot 1 \end{array} \right.$ The Unciæ of the fourth Power.

Add $\left\{ \begin{array}{c} 1 \cdot 5 \cdot 10 \cdot 10 \cdot 5 \cdot 1 \\ \quad 1 \cdot 5 \cdot 10 \cdot 10 \cdot 5 \cdot 1 \end{array} \right.$ Unciæ of the fifth Power.

Add $\left\{ \begin{array}{c} 1 \cdot 6 \cdot 15 \cdot 20 \cdot 15 \cdot 6 \cdot 1 \\ \quad 1 \cdot 6 \cdot 15 \cdot 20 \cdot 15 \cdot 6 \cdot 1 \end{array} \right.$ Unciæ of the 6th Power.

$1 \cdot 7 \cdot 21 \cdot 35 \cdot 35 \cdot 21 \cdot 7 \cdot 1$ Unciæ of 7th Pow.

And so on in this manner *ad infinitum*.

Now if these Numbers are prefixed to the aforesaid Letters, all the Terms will be compleated with their respective Unciæ, and will stand thus;

$$a^7 + 7a^6b + 21a^5b^2 + 35a^4b^3 + 35a^3b^4 + 21a^2b^5 + 7ab^6 + b.$$

But that the Business of finding these Unciæ, may be rendered yet more easy for Practice, it will be convenient to consider what Series or Progression, the Unciæ of each Term do make from the aforesaid additions.

Unciæ of the first Term.	Unciæ of the second Term.	Unciæ of the third Term.	Unciæ of the fourth Term.	Unciæ of the fifth Term.	Unciæ of the sixth Term.	Unciæ of the seventh Term.	Unciæ of the eighth Term, &c.	
1	1	1	1	1	1	1	1	Unciæ of the single Quantities.
1	2	1	1	1	1	1	1	Unciæ of the Square.
1	3	3	1	1	1	1	1	Unciæ of the Cube.
1	4	6	4	1	1	1	1	Unciæ of the 4th Power.
1	5	10	10	5	1	1	1	Unciæ of the 5th Power.
1	6	15	20	15	6	1	1	Unciæ of the 6th Power.
1	7	21	35	35	21	7	1	Unciæ of the 7th Power, &c.

The Unciæ of the first Term are only a Series of Units, whose Sum is every where the Unciæ of the second Term. The Unciæ of the second Term, are a Series of Numbers in Arithmetick Progression, whose Sum is every where the Unciæ of the next Superior Power in the third Term, and may be found by Proposi-

tion

tion 1. Chap. 6. Part 1. For Instance, in the seventh Power it will be $\frac{0+1 \times 6}{2} = 21 =$ the Uncia of the third Term.

The rest of the Unciæ are a compounded Series, whose respective Sums may be obtained from the Unciæ of their precedent Terms.

Thus $\frac{21 \times 5}{3} = 35$. Then $\frac{35 \times 4}{4} = 35$. Again $\frac{35 \times 3}{5} = 21$.

And $\frac{21 \times 2}{6} = 7$ &c.

From hence may be deduced this general Rule.

R U L E.

If the Index of the first Letter of any Term be multiplied into it's own Uncia, and that Product be divided by the Number of Terms to that Place; the Quotient will be the Uncia of the next succeeding Term forward.

That is, by the help of those Indices that belong to the several Powers of the first or leading Letter only (as a) the true Unciæ of every Term may be easily understood.

E X A M P L E 2.

Let it be required to compleat all the Terms of the aforesaid several Powers, viz. $a^7 + a^6 b + a^5 b^2 + a^4 b^3 + a^3 b^4 + a^2 b^5 + a b^6 + b^7$, with their proper Unciæ.

1. The Index of a^7 the first Term will be the Uncia of the second Term. Thus $a^7 + 7 a^6 b$.

2. Then half the second Term's Index into it's Uncia, viz. $\frac{7 \times 6}{2} = 21$ will be the third Term's Uncia. Thus $a^7 + 7 a^6 b + 21 a^5 b^2$ will be the three first Terms.

3. Again $\frac{21 \times 5}{3} = 35$ is the Uncia of the fourth Term, whence $a^7 + 7 a^6 b + 21 a^5 b^2 + 35 a^4 b^3$ will be the four first Terms.

4. And $\frac{35 \times 4}{4} = 35$ will be the Uncia of the fifth Term, whence $a^7 + 7 a^6 b + 21 a^5 b^2 + 35 a^4 b^3 + 35 a^3 b^4$ will be the five first Terms.

And so proceed 'till all the Terms are compleated with their respective Unciæ; which will stand, thus $a^7 + 7 a^6 b + 21 a^5 b^2 + 35 a^4 b^3 + 35 a^3 b^4 + 21 a^2 b^5 + 7 a b^6 + b^7$.

Now

Now here it may be further observed, that the *Unciæ* do only increase until the Indices of the two Letters become equal, or change Places; and then the rest of the *Unciæ* will return or decrease in the same order. That is, wherever the Indices of the Letters are alike, there the *Unciæ* will be alike.

And therefore one needs to find the *Unciæ* (as before) but to half the Number of Terms in any Power.

If what hath been said, and the Work of the *Example* be well understood, I presume it will be found very easy to raise any Power from a Binomial or Residual Root, to what Height you please; without the Trouble of a continued Involution; and without the Help of such a Table of Powers as is proposed by Mr *Oughtred* in his Key to the *Mathematicks*, Page 40, and since by others.

Now from these Considerations it was, that I proposed this thod of raising Powers in my Compendium of *Algebra*, Page 57, as wholly New (*viz.* so much of it as was there useful) having then (I profess) neither seen the Way of doing it, nor so much as heard of it's being done. But since the writing of that Tract, I find in Dr *Wallis's History of Algebra*, Page 319 and 331, that the Learned Sir *Isaac Newton* had discovered it long before: which the Doctor sets down in this manner.

Let m be the Exponent of the Power.

$$\text{Then } \left\{ 1 \times \frac{m-0}{1} \times \frac{m-1}{2} \times \frac{m-2}{3} \times \frac{m-3}{4} \times \frac{m-4}{5} \text{ \&c.} \right.$$

Will be the Series of the *Unciæ* required; but he doth not tell us how they first came to be found out, nor have I ever met with the least Hint of it in any Author.

Sect. 6. Evolution of whole Quantities.

Evolution is the extracting of Roots from any given Power. That is, it is the Converse Work to that of Involution, and in single Quantities it is easy, if the given Power have such a Root as is required, which may be thus known.

If the given Power have no Numbers prefixed to it, and it's Index can be divided by the Index of the Root required, the Quotient will be the Index of the Root sought. Thus, if the Cube Root of $a a a a a a$, *viz.* a^6 were required (the Index of the

Cube is 3) then $3 \overline{) 6}$ (2. That is, $a^{\frac{6}{3}} = a^2$ the Root required. And such Operations are usually set down

Thus

Thus	1	a^6	$a^6 b^6$	$a^6 b^6 d^6$
1 $\sqrt{w^2}$	2	a^3	$a^3 b^3$	$a^3 b^3 d^3$
1 $\sqrt{w^3}$	3	a^2	$a^2 b^2$	$a^2 b^2 d^2$
3 $\sqrt{w^2}$	4	a	$a b$	$a b d$

Note, The Figures placed in the Margin after the Sign ($\sqrt{w}$) of Evolution, denote the Index of the Root to be extracted.

If the given Powers have Co-efficients: (*viz.* Numbers prefixed to them;) then you must extract their respective Roots, as in Vulgar *Arithmetick*.

Thus	1	$81 a^4$	$1296 a^3 b^3$	$20736 a^4 b^4 c^4$
1 $\sqrt{w^2}$	2	$9 a^2$	$36 a^4 b^4$	$144 a^2 b^2 c^2$
1 $\sqrt{w^4}$	3	$3 a$	$6 a^2 b^2$	$12 a b c$
or 2 $\sqrt{w^2}$	4	$3 a$	$6 a^2 b^2$	$12 a b c$

But if the Root required cannot be truly extracted out of both the Co-efficients and Indices of the given Power; then it is a *Surd*, and must have the Sign of the Root required prefixed to it.

Thus	1	a^5	$67 a^4$	$216 b b b d d d$
1 $\sqrt{w^2}$	2	$\sqrt{a^5}$	$\sqrt{67 a^4}$	$\sqrt{216 b b b d d d}$
1 $\sqrt{w^3}$	3	$\sqrt[3]{a^5}$	$\sqrt[3]{67 a^4}$	$6 b d$

Evolution of compound Quantities or Powers, is a little more troublesome than that of Single Powers; and would require a great many Words to explain the Manner and Reason of forming the several Canons, that are commonly used in extracting the Roots of compound Quantities; especially if the Powers be very high, &c. I shall therefore for Brevity's sake omit them, and instead thereof propose an easy Method of discovering the Roots of all compound Powers in general. And in order to that, it will be necessary to premise; that if either the Sum or Difference of several Quantities be involved to any Power, there will arise so many single Powers of the same height, as there are different Quantities.

As for instance, if $a + b + d$ be squared, that is, be involved to the second Power, it will be $aa + 2ab + 2ad + bb + 2bd + dd$, here you have aa , bb , and dd . Again, if $a + b + d$ were cubed, *viz.* involved to the third Power, then you will have aaa , bbb , ddd , in it, &c.

Y

Whence

Whence it follows that in extracting the Roots of all compound Quantities, there must be considered,

1. How many different Letters (or Quantities) there are in the given Power.

2. Whether the single Powers of each of those Letters be of an equal Height, and have in them such a single Root as is required: which if they have, extract it as before.

3. Connect those single Roots together with the Sign $+$, and involve them to the same Height with the given Power; that being done, compare the new raised Power with the given Power; and if they are alike in all their respective Terms, then you have the Root required; or if they differ only in their Signs, the Root may be easily corrected with the Sign $-$ as occasion requires.

Example 1. Let be required to extract the Square Root of $cc + 2cb - 2cd + bb - 2bd + dd$. In this Compound Square, there are three distinct Powers, viz. bb , cc , dd , whose single Roots are b , c , d , wherefore I suppose the Root sought to be $b + c + d$, or rather $b + c - d$, because in the given Power there is $-2cd$, and $-2bd$, therefore I conclude it is $-d$; then $b + c - d$, being squared, produces $bb + 2bc - 2bd + cc - 2cd + dd$, which I find to be the same in all it's Terms with the given Power, although they stand in a different Position; consequently $b + c - d$ is the true Root required.

Example 2. It is required to extract the Square Root of $a^4 - 2aabb + b^4$. Here are but two single Powers, viz. a^4 and b^4 , whose Square Roots are aa , and bb . And because in the given Power there is $-2aabb$, therefore I conclude it must either be $aa - bb$, or $bb - aa$. Both which, being involved, will produce $a^4 - 2aabb + b^4$; consequently the Root sought may either be $aa - bb$, or $bb - aa$, according to the Nature or Design of the Question from whence the given Power was produced.

Example 3. Let it be required to extract the Square Root of $36aaaa + 108aa + 81$. Here the two single Powers are $36aaaa$, and 81 , whose Roots are $6aa$ and 9 . And because the Signs are all $+$, therefore I suppose the Root to be $6aa + 9$, the which being involved doth produce $36a^4 + 108aa + 81$; consequently $6aa + 9$ is the true Root required.

Example 4. Suppose it were required to extract the Cube Root of $125aaa + 300aae - 450aa + 250aee - 720ae + 64eee + 540a - 288ee + 432e - 216$. In this Example there are three distinct Powers, viz. $125aaa$, $64eee$, and -216 . And the Cube Root of $125aaa$ is $5a$; of $64eee$ is $4e$; of -216 is -6 . Wherefore I suppose the Root sought to be $5a + 4e - 6$, which being involved to the third Power, does produce

*

the

the same with the given Power; consequently $5a + 4e - 6$ is the Cube Root required.

But if the new Power, raised from the supposed Root (being involved to it's due Height), should not prove the same with the given Power, *viz.* if it hath either more or fewer Terms in it, &c. then you may conclude the given Power to be a Surd, which must have it's proper Sign prefixed to it, and cannot be otherwise expressed, until it come to be envolved in Numbers.

Example 5. Suppose it were required to extract the Cube Root of $27aaa + 54baa + 8bbb$. Here are two distinct and perfect Cubes, *viz.* $27aaa$, and $8bbb$, whose Cube Roots are $3a$ and $2b$. Wherefore one may suppose the Root sought to be $3a + 2b$, which being involved to the third Power, is $27aaa + 54baa + 36bba + 8bbb$. Now this new raised Power hath one Term (*viz.* $36bba$) more in it than the given Power hath; but this being a perfect Cube, one may therefore conclude the given Power is not so, *viz.* it is a Surd, and hath not such a Root as was required, but must be expressed, or set down,

$$\text{Thus } \sqrt[3]{27aaa + 54baa + 8bbb}.$$

If these Examples be well understood, the Learner will find it very easy by this Method of proceeding to discover the true Root of any given Power whatsoever.

C H A P. III.

Of Algebraick Fractions, or Broken Quantities.

Sect. I. Notation of Fractional Quantities.

F*Ractional Quantities* are expressed or set down like Vulgar Fractions in common *Arithmetick*.

$$\text{Thus } \left\{ \begin{array}{l} \frac{a}{b}, \frac{2bc}{d}, \frac{5b - 4a}{4d + 7b} \end{array} \right. \begin{array}{l} \text{Numerators.} \\ \text{Denominators.} \end{array}$$

How they come to be so, see *Case 4*, in the last Chapter of *Division*. These Fractional Quantities are managed in all respects like Vulgar Fractions in Common *Arithmetick*.

Sect. 2. *To alter or change different Fractions into one Denomination, retaining the same Value.*

R U L E.

MULTIPLY all the Denominators into each other for a new Denominator, and each Numerator into all the Denominators but it's own for new Numerators.

E X A M P L E S.

Let it be required to bring $\frac{a}{b}$ and $\frac{d}{c}$ into one Denomination.

First $a \times c$, and $d \times b$, will be the Numerators, and $b \times c$ will be the common Denominator, viz. $\frac{ca}{bc}$ and $\frac{bd}{bc}$ are the two Fractions required: that is, $\frac{ca}{bc} = \frac{a}{b}$, and $\frac{bd}{bc} = \frac{d}{c}$.

Again, let $\frac{b+c}{a+b}$ and $\frac{d-e}{b-d}$ be brought in one Denomination, and they will be $\frac{bb+bc-bd-de}{ba+bb-da-bd}$ and $\frac{ad-ac+bd-be}{ba+bb-da-bd}$ &c.

Sect. 3. *To bring whole Quantities into Fractions of a given Denomination.*

R U L E.

MULTIPLY the whole Quantities into the given Denominator for a Numerator, under which subscribe the given Denominator, and you will have the Fraction required.

E X A M P L E S.

Let it be required to bring $a+b$ into a Fraction, whose Denominator is $d-a$. First $a+b \times d-a$ is $da+bd-aa-ba$. Then $\frac{da+bd-aa-ba}{d-a}$ is the Fraction required.

Again $b + \frac{aa}{d}$ will be $\frac{db+a}{d}$. And $\frac{aa}{d} - a$ will be $\frac{aa-da}{a}$.

Also $a+b + \frac{aa+bb}{a-b}$ will be $\frac{2aa}{a-b}$.

When

When whole Quantities are to be set down Fraction-wise, subscribe an Unit for the Denominator. Thus ab is $\frac{ab}{1}$. And $aa - bb$, is $\frac{aa - bb}{1}$, &c.

Sect. 4. To abbreviate, or reduce Fractional Quantities into their lowest Denomination.

R U L E.

DIVIDE both the Numerator and Denominator by their greatest common Divisor, viz. by such Quantities as are found in both; and their Quotients will be the Fraction in it's lowest Term.

Thus $\frac{aac}{dc}$ is $\frac{aa}{d}$. $\frac{abbb}{abc}$ is $\frac{bb}{c}$. And $a + \frac{bdc}{bc} = a + d$.

In such single Fractions as these, the common Divisors (if there be any) are easily discovered by Inspection only; but in compound Fractions it often proves very troublesome, and must be done either by dividing the Numerator by the Denominator, until nothing remains, when that can be done: or else finding their common Measure, by dividing the Denominator by the Numerator, and the Numerator by the Remainder, and so on, as in Vulgar Fractions. (Sect. 4. Page 51.)

E X A M P L E S.

Suppose $\frac{aac - aad}{cd - dd}$ were to be reduced lower.

Then $cd - dd \overline{)aac - aad}$ $\left(\frac{aa}{d}\right)$ the Fraction required.

In this Example it so happens that the Numerator is divided just off by the Denominator; but in the next it is otherwise, and requires a double Division to find out the common Measure, viz.

Let it be required to reduce $\frac{aaa - abb}{aa + 2ab + bb}$ to it's lowest Terms.

First $aa + 2ab + bb \overline{)aaa - abb}$ $(a$
 $aaa + 2aab + abb$
 $- 2aab - 2abb$ the Remainder.

Then $-2aab - 2abb \overline{)aa + 2ab + bb}$ $\left(-\frac{1}{2b} - \frac{1}{2a}\right)$
 $aa + ab$
 $ab + bb$
 $ab + bb$
 $0 \quad 0$

Hence

Hence it appears that $-2aab - 2abb$ is the common Measure; by which $aaa - abb$ being divided,

$$\begin{array}{r} \text{Viz. } -2aab - 2abb \overline{) aaa - abb} \left(-\frac{a}{2b} + \frac{1}{2} \right. \\ \underline{aaa + aab} \\ -aab - abb \\ \underline{-aab - abb} \\ 0 \end{array}$$

Then $-\frac{a}{2b} + \frac{1}{2}$ is the new Numerator; and $-\frac{1}{2b}$ is the new Denominator. But $-\frac{a}{2b} + \frac{1}{2} = \frac{-2a + 2b}{4b}$
 $= \frac{-a + b}{2b}$ the Numerator; and $-\frac{1}{2b} - \frac{1}{2a} = \frac{-2a - 2b}{4ba}$
 $= \frac{-a - b}{2ba}$ the Denominator. Let both be multiplied with $2ba$, and you will have $\frac{-aa + ab}{-a - b}$ the Numerator. Or changing the Signs of all the Quantities, it will be $\frac{aa - ab}{a + b}$ the new Fraction required. That is, $\frac{aa - ab}{a + b} = \frac{aaa - abb}{aa + 2ab + bb}$.

Again, let it be required to reduce $\frac{dd - bb}{ddd - bbb}$.

The common Measure of this Fraction will be the easiest found (as appears from Trials) by dividing the Denominator by the Numerator, &c. Thus,

$$\begin{array}{r} dd - bb \overline{) ddd - bbb} (d \\ \underline{ddd - bbd} \\ +bbd - bbb \overline{) dd - bb} \left(\frac{d}{bb} \right. \\ \underline{dd - bd} \\ +bd - bb \overline{) bbd - b^3} \\ \underline{bbd - b^3} \\ 0 \end{array}$$

Hence it appears that $bd - bb$ is the common Measure that will divide both the Numerator and the Denominator.

Consequently

Consequently $bd - bb) dd - bb \left(\frac{d}{b} + 1, \text{ is the new Numerator.} \right.$

$$\begin{array}{r} dd - bb \\ + db - bb \\ \hline db - bb \\ \hline 0 \quad 0 \end{array}$$

And $bd - bb) ddd - bbb \left(\frac{dd}{b} + d + b \text{ the new Denominator.} \right.$

$$\begin{array}{r} ddd - bbb \\ + ddb - bbb \\ \hline ddb - bbb \\ \hline + bbd - bbb \\ \hline bbd - bbb \\ \hline 0 \quad 0 \end{array}$$

Let both be multiplied with b , and then you will have
 $\frac{d + b}{dd + bd + bb}$ the Numerator, } of the Fraction required.
the Denominator,

But if after all Means used (as above) there cannot be found one common Measure to both the Numerator and Denominator; then is that Fraction in it's least Terms already.

Note, These Operations will be understood by a Learner after he hath passed thro' *Multiplication*, and *Division* of Fractions.

Sect. 5. Addition and Subtraction of Fractional Quantities.

THE given Fractions being of one Denomination, or if they are not, make them so, per Sect. 4. Then,

R U L E.

Add or subtract their Numerators, as Occasion requires, and to their Sum or Difference, subscribe the common Denominator: as in Vulgar Fractions.

Examples in Addition.

1	$\frac{bb}{c}$	$\frac{a+b}{d}$	$\frac{2a-b}{d+c}$	$\frac{a-b+d}{d+a}$
2	$\frac{aa}{c}$	$\frac{2a+c}{d}$	$\frac{2b-a}{d+c}$	$\frac{a+b-d}{d+a}$
1+2	$\frac{bb+aa}{c}$	$\frac{3a+b+c}{d}$	$\frac{a+b}{d+c}$	$\frac{2a}{d+a}$

Examples

Examples in Subtraction.

1-2	1	$\frac{bb+aa}{c}$	$\frac{a+b}{d+c}$	$\frac{3a+b+c}{d}$	$\frac{2b}{d+a}$
	2	$\frac{bb}{c}$	$\frac{2b-a}{d+c}$	$\frac{2a+c}{d}$	$\frac{a+b-d}{d+a}$
	3	$\frac{aa}{c}$	$\frac{2a-b}{d+c}$	$\frac{a+b}{d}$	$\frac{b-a+d}{d+a}$

Sect. 6. *Multiplication of Fractional Quantities.*

FIRST prepare mixed Quantities (if there be any) by making them improper Fractions, and whole Quantities by subscribing an Unit under them; as per Sect. 3. Then,

R U L E.

Multiply the Numerators together for a new Numerator, and the Denominators together for a new Denominator; as in Vulgar Fractions.

Thus

1	$\frac{ab}{c}$	$\frac{3a-2b}{2d+c}$
	$\frac{d}{f}$	$\frac{4a+2b}{d}$
1 x 2	3	$\frac{abd}{cf} \quad \frac{12aa-2ab-4bb}{2dd+dc}$

Suppose it were required to multiply $2a + \frac{b}{c} - 25$, with $3b + 4c$. These prepared for the Work (per Sect. 3.) will stand

Thus

1	$\frac{2ac + b - 25c}{c}$
	$\frac{3b + 4c}{1}$
2 x 2	3 $\frac{6bac + 3bb - 75bc + 8acc + 4bc - 100cc}{c}$
or	4 $6ba - 71b + 8ac - 100c + \frac{3bb}{c}$ per Sect. 4.

N. B. Any Fraction is multiplied with it's Denominator by casting off, or taking the Denominator away. Thus $\frac{b}{a} \times a$ gives b . For $\frac{b}{a} \times \frac{a}{1} = \frac{ba}{a} = b$, &c.

Sect. 7. Division of Fractional Quantities.

THE Fractional Quantities being prepared, as directed in the last Section. Then,

R U L E.

Multiply the Numerator of the Dividend, into the Denominator of the Divisor, for a new Numerator; and multiply the other two together for a new Denominator; as in Vulgar Fractions.

E X A M P L E S.

Let $\frac{abd}{cf}$ be divided by $\frac{ab}{c}$, the Work may stand thus,

$$\frac{ab}{c} \bigg) \frac{abd}{cf} \left(\frac{abdc}{abcf} = \frac{d}{f} \text{ per Sect. 4.} \right.$$

$$\text{Or thus } \begin{array}{l|l|l|l} 1 & \frac{abd}{cf} & \frac{a+b}{d} & \frac{aaa-bbb}{a+b} \\ 2 & \frac{ab}{c} & \frac{c-b}{a} & \frac{aa-ab+bb}{c} \\ \hline 1 \div 2 & 3 & \frac{d}{f} & \frac{aa+ba}{dc-db} \end{array} \quad \begin{array}{l} \\ \\ \hline \frac{aaac-bbbc}{aaa+bbb} \end{array}$$

Suppose it were required to divide $aa + \frac{3abb}{a+4b}$ by $a+b$.

The Work prepared will stand thus,

$$\frac{a+b}{1} \bigg) \frac{aaa+4aab+3abb}{a+4b} \left(\frac{aaa+4aab+3abb}{aa+5ba+4bb} \right. \text{ But } \frac{aaa+4aab+3abb}{aa+5ba+4bb} = \frac{aa+3b}{a+4b} \text{ (per Sect. 4) }$$

When Fractions are of one Denomination, cast off the Denominators, and divide the Numerators. Thus, if $\frac{ab^3}{c}$ were to be divided by $\frac{bb}{c}$ it will be $bb)ab^3$ (ab the Quotient required.

For $\frac{bb}{c} \frac{ab^3}{c} \left(\frac{ab^3c}{bb^3c} \right)$. But $\frac{ab^3c}{bb^3c} = ab$ (per Sect. 4.)

Again, suppose it were required to divide $\frac{a^3 - abb}{c - d}$ by $\frac{aa + 2ab + bb}{c - d}$. Casting off $c - d$ in both, it will be $aa + 2ab + bb) aaa - abb \left(\frac{aa - ba}{a + b}, \&c. \right.$

Sect. 8. Involution of Fractional Quantities.

R U L E.

INVOLVE the Number into itself for a new Numerator, and the Denominator into itself for a new Denominator; each as often as the Power requires.

Thus	1	$\frac{b}{a}$	$\frac{3bc}{2ad}$	$\frac{b+d}{a-c}$
$I \odot^2$	2	$\frac{bb}{aa}$	$\frac{9bbcc}{4aadd}$	$\frac{bb + 2bd + dd}{aa - 2ac + cc}$
$I \odot^3$	3	$\frac{bbb}{aaa}$	$\frac{27bbccc}{8aaaddd}$	$\frac{bbb + 3bbd + 3bdd + ddd}{aaa - 3aac + 3acc - ccc}$

Sect. 9. Evolution of Fractional Quantities.

IF the Numerator and Denominator of the Fraction have each of them such a Root as is required (which very rarely happens) then evolve them; and their respective Roots will be the Numerator and Denominator of the new Fraction required.

Thus	1	$\frac{9aabb}{4dd}$	$\frac{aa + 2ab + bb}{aa - 2ab + bb}$
$I \omega^2$	2	$\frac{3ab}{2d}$	$\frac{a+b}{a-b}$
Again	1	$\frac{27aaabbb}{8ddd}$	$\frac{aaa + 3aab + 3abb + bbb}{aaa - 3aab + 3abb - bbb}$
$I \omega^3$	2	$\frac{3ab}{2d}$	$\frac{a+b}{a-b}$

Sometimes it so falls out, that the Numerator may have such a Root as is required, when the Denominator hath not; or the Denominator

minator may have such a Root, when the Numerator hath not. In those Cases the Operations may be set down

Thus	1	$\frac{a a b b}{d d d}$	$\frac{a a a + 4 b b - d d}{a a - 2 a b + b b}$
$I w^2$	2	$\frac{a b}{\sqrt{d d d}}$	$\frac{\sqrt{a a a + 4 b b - d d}}{a - b}$

But when neither the Numerator, nor the Denominator have just such a Root as is required, prefix the radical Sign of the Root to the Fraction; and then it becomes a Surd, as in the last Step; which brings me to the Business of managing Surds.

CHAP. IV.

Of Surd Quantities.

THE whole Doctrine of Surds (as they call it). were it fully handled, would require a very large Explanation (to render it but tolerably intelligible); even enough to fill a Treatise itself, if all the various Explanations that may be of Use to make it easy should be inserted; without which it is very intricate and troublesome for a Learner to understand. But now these tedious Reductions of Surds, which were heretofore thought useful to fit Equations for such a Solution, as was then understood, are wholly laid aside as useless: Since the new Methods of resolving all sorts of Equations render their Solutions equally easy, although their Powers are never so high. Nay, even since the true Use of Decimal *Arithmetick* hath been well understood, the Business of Surd Numbers has been managed that Way; as appears by several Instances of that Kind in Dr *Wallis's History of Algebra*, from Page 23, to 29.

I shall therefore, for Brevity sake, pass over those tedious Reductions, and only shew the young *Algebraist* how to deal with such Surd Quantities as may arise in the Solution of hard Questions.

Sect. 1. Addition and Subtraction of Surd Quantities.

Case 1. **W**HEN the Surd Quantities are Homogeneous, (*viz.* are alike) add or subtract the rational Part, if they

are joined to any, and to their Sum, or Difference, adjoin the irrational or Surd.

Examples in Addition.

$1+2$	1	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
	3	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$

$1+2$	1	$4d^3\sqrt{aa}$	$b+^3\sqrt{aa-cc}$	$bc:^5\sqrt{aa+d}$
	2	$d^3\sqrt{aa}$	$c-^3\sqrt{aa-cc}$	$3bc:^5\sqrt{aa+d}$
	3	$5d^3\sqrt{aa}$	$b+c$	$4bc:^5\sqrt{aa+d}$

Examples in Substraction.

$1-2$	1	$12\sqrt{bc}$	$10b\sqrt{ac}$	$4b\sqrt{aa+cc}$
	2	$7\sqrt{bc}$	$4b\sqrt{ac}$	$3b\sqrt{aa+cc}$
	3	$5\sqrt{bc}$	$6b\sqrt{ac}$	$b\sqrt{aa+cc}$

$1-2$	1	$5d:^3\sqrt{aa}$	$b+c$	$4bc:^5\sqrt{aa+d}$
	2	$4d:^3\sqrt{aa}$	$c-^3\sqrt{aa-cc}$	$3bc:^5\sqrt{aa+d}$
	3	$d:~^3\sqrt{aa}$	$b+^3\sqrt{aa-cc}$	$bc:~^5\sqrt{aa+d}$

Case 2. When the Surd Quantities are Heterogeneous, (*viz.* their Indices are unlike) they are only to be added, or subtracted by their Signs, *viz.* + or -. And from thence will arise Surds either Binomial, or Residual.

Examples in Addition.

$1+2$	1	$\sqrt{bc}$	$4d\sqrt{a}$	$^3\sqrt{ac-ba}$
	2	$\sqrt{ba}$	$3b\sqrt{ac}$	$\sqrt{ac+ba}$
	3	$\sqrt{bc}+\sqrt{ba}$	$4d\sqrt{a}+3b\sqrt{ac}$	$^3\sqrt{ac-ba}+\sqrt{ac+ba}$

Examples in Substraction.

$1-2$	1	$\sqrt{bc}$	$b-d\sqrt{aaa+ca}$
	2	$\sqrt{ba}$	$d-2a\sqrt{bd+dd}$
	3	$\sqrt{bc}-\sqrt{ba}$	$b-d\sqrt{aaa+ca}-d+2a\sqrt{bd+dd}$

Sect.

Sect. 2. *Multiplication of Surd Quantities.*

Case 1. **W**HEN the Quantities are pure Surds of the same Kind; multiply them together, and to their Product prefix their radical Sign.

E X A M P L E S.

1 × 2	1	$\sqrt{b}$	$\sqrt{ba + da}$	$\sqrt{aa + bb}$
	2	$\sqrt{a}$	$\sqrt{ca}$	$\sqrt{aa - bb}$
	3	$\sqrt{ba}$	$\sqrt{bcaa + dcaa}$	$\sqrt{aaaa - bbbb}$

Case 2. If Surd Quantities of the same Kind (as before) are joined to rational Quantities, then multiply the rational into the rational; and the Surd into the Surd, and join their Products together.

E X A M P L E S.

1 × 2	1	$d\sqrt{bc}$	$5cd\sqrt{ba + da}$	$15\sqrt{ab}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{a}$
	3	$3db\sqrt{bca}$	$15cda\sqrt{bcaa - dcaa}$	$75\sqrt{abd}$

Sect. 6. *Division of Surd Quantities.*

Case 1. **W**HEN the Quantities are pure Surds of the same Kind, and can be divided off, (*viz.* without leaving a Remainder) divide them, and to their Quotient prefix their radical Sign.

E X A M P L E S.

1 ÷ 2	1	$\sqrt{ba}$	$\sqrt{bcaa + dcaa}$	$\sqrt{aaaa - bbbb}$
	2	$\sqrt{b}$	$\sqrt{ca}$	$\sqrt{aa - bb}$
	3	$\sqrt{a}$	$\sqrt{ba + da}$	$\sqrt{aa + bb}$

Case 2. If Surd Quantities, of the same Kind, are joined to rational Quantities; then divide the rational by the rational, if it can be, and to their Quotient join the Quotient of the Surd divided by the Surd with it's first radical Sign.

E X A M P L E S.

1 ÷ 2	1	$3db\sqrt{bca}$	$15cda\sqrt{bcaa + dcaa}$	$75\sqrt{abd}$
	2	$3b\sqrt{a}$	$3a\sqrt{ca}$	$5\sqrt{d}$
	3	$d\sqrt{bc}$	$5cd\sqrt{ba + da}$	$15\sqrt{ab}$

Note,

Note, If any Square be divided by it's Root, the Quotient will be it's Root.

E X A M P L E S.

$$\begin{array}{l|l|l|l}
 \text{I} & a & bb + 2bc + cc & aaaa - 2bbba + bbbb \\
 \hline
 2 & \sqrt{a} & \sqrt{bb + 2bc + cc} & \sqrt{a^4 - 2bbba + b^4} \\
 \hline
 \text{I} \div 2 & 3 & \sqrt{a} & \sqrt{bb + 2bc + cc} \quad \sqrt{a^4 - 2bbba + b^4}
 \end{array}$$

Sect. 4. *Involution of Surd Quantities.*

Case 1. **W**HEN the Surds are not joined to rational Quantities; they are involved to the same Height as their Index denotes, by only taking away their radical Sign.

E X A M P L E S.

$$\begin{array}{l|l|l|l|l}
 \text{I} \odot^2 & \text{I} & \sqrt{a} & \sqrt{bca} & \sqrt{aa - bb} \\
 \hline
 & 2 & a & bca & aa - bb
 \end{array}$$

Case 2. When the Surds are joined to rational Quantities; involve the rational Quantities to the same Height as the Index of the Surd denotes; then multiply those involved Quantities into the Surd Quantities, after their radical Sign is taken away, as before.

E X A M P L E S.

$$\begin{array}{l|l|l|l|l}
 \text{I} \odot^2 & \text{I} & b\sqrt{a} & 5d\sqrt{ca} & 3b\sqrt{aa - dd} \\
 \hline
 & 2 & bba & 25ddca & 9bbba - 9bbdd \\
 \hline
 \text{I} \odot^3 & \text{I} & a: \sqrt[3]{bc} & 3d: \sqrt[3]{aa + bb} & da: \sqrt[3]{b} \\
 \hline
 & 2 & a a a b c & 27 d d d a a + 27 d d d b b & d d d a a b
 \end{array}$$

The Reason of only taking away the radical Sign, as in *Case 1.* is easily conceived, if you consider that any Root being involved into itself, produces a Square, &c. And from thence the Reason of those Operations performed by the second Case may be thus stated.

Suppose $b\sqrt{a} = x$. Then $\sqrt{a} = \frac{x}{b}$ per *Axiom 4.* and both Sides of the Equation being equally involved, it will be $a = \frac{xx}{bb}$. Then multiplying both Sides of the Equation into bb , it becomes $bba = xx$ per *Axiom 3.* Which was to be proved.

Again,

Again, Let $5d\sqrt{ca} = x$: Then $\sqrt{ca} = x \frac{x}{5d}$, and ca
 $= \frac{xx}{25dd}$.

Also from hence it will be easy to deduce the Reason of multiplying **Surd** Quantities, according to both the *Cases*. For

Suppose {	1	$\sqrt{b} = z$	} Example 1. Case 1.
	2	$\sqrt{a} = x$	
1 $\odot^2$	3	$b = zz$	
2 $\odot^2$	4	$a = xx$	
3 $\times$ 4	5	$ba = z z x x$. per Axiom 2.	
5 $\cup \cup^2$	6	$\sqrt{ba} = zx$. which was to be proved.	

Let {	1	$d\sqrt{bc} = z$	} Example 1. Case 2.
	2	$3b\sqrt{a} = x$	
1 $\div d$	3	$\sqrt{bc} = \frac{z}{d}$	
2 $\div 3b$	4	$\sqrt{a} = \frac{x}{3b}$	
4 $\times$ 3	5	$\sqrt{abc} = \frac{zx}{3bd}$, from what is proved above.	
5 $\times 3bd$	6	$3bd\sqrt{bca} = zx$, &c. for the rest.	

Division being the Converse to *Multiplication*, needs no other Proof.

CHAP. V.

Concerning the Nature of Equations and how to prepare them for a Solution.

WHEN any Problem or Question is proposed to be analytically resolved; it is very requisite that the true Design or Meaning thereof, be fully and clearly comprehended (in all it's Parts) that so it may be truly abstracted from such ambiguous Words as Questions of this Kind are often disguised with; otherwise it will be very difficult, if not impossible, to state the Question right in it's substituted Letters, and ever to bring it to an Equation by such various Methods of ordering those Letters as the Nature of the Questions may require.

Now

Now the Knowledge of this difficult Part of the Work is only to be obtained by Practice, and a careful minding the Solution of such leading Questions as are in themselves very easy. And for that Reason I have inserted a Collection of several Questions; wherein there is great Variety.

Having got so clear an Understanding of the Question proposed, as to place down all the Quantities concerned in their due Order, viz. all the substituted Letters, in such Order as their Nature requires; the next thing must be to consider whether it be limited or not. That is, whether it admits of more Answers than one. And to discover that, observe the two following Rules.

R U L E 1.

When the Number of the Quantities sought exceed the Number of the given Equations, the Question is capable of innumerable Answers.

E X A M P L E.

Suppose a Question were proposed thus; there are three such Numbers, that if the first be added to the second, their Sum will be 22. And if the second be added to the third, their Sum will be 46. What are those Numbers?

Let the three Numbers be represented by three Letters, thus call the first a , the second e , and the third y .

Then $\left\{ \begin{array}{l} a + e = 22 \\ e + y = 46 \end{array} \right\}$ according to the Question.

Here the Number of Quantities sought are three, a , e , y , and the Number of the given Equations are but two. Therefore this Question is not limited, but admits of various Answers; because for any one of those three Letters you may take any Number at Pleasure, that is less than 22. Which with a little Consideration will be very easy to conceive.

R U L E 2.

When the Number of the given Equations (not depending upon one another) are just as many as the Number of the Quantities sought; then is the Question truly limited, viz. each Quantity sought hath but one single Value.

As for Instance, let the aforesaid Question be proposed thus. There are three Numbers (a , e , and y , as before); if the first be added to the second, their Sum will be 22; if the second be added

to

to the third, their Sum will be 46; and if the first be added to the third, their Sum will be 36. What are the Numbers? That is, $a + e = 22$. $e + y = 46$. and $a + y = 36$. Now the Question is perfectly limited, each single Quantity having but one single Value, to wit $a = 6$, $e = 16$, and $y = 30$.

N. B. If the Number of the given Equations exceeds the Number of the Quantities sought; they not only limit the Question, but oftentimes render it impossible, by being proposed inconsistent one to another.

Having truly stated the Question in it's substituted Letters, and found it limited to one Answer (or at least so bounded as to have a certain determinate Number of Answers), then let all those substituted Letters be so ordered or compared together, either by adding, subtracting, multiplying, or dividing them, &c. according as the Nature of the Question requires, until all the unknown Quantities except one, are cast off or vanished; but therein great Care must be taken to keep them to an exact Equality; and when that unknown Quantity, or some Power of it (as Square, Cube, &c.) is found equal to those that are known; then the Question is said to be brought to an Equation, and consequently to a Solution, viz. fitted for an Answer.

But no particular Rules can be prescribed for the casting off, or getting away Quantities out of an Equation; that Part of the Art is only to be obtained by Care and Practice. And when that is done, it generally happens so, that the unknown Quantity which is retained in the Equation, is so mixed and entangled with those that are known, that it often requires some Trouble and Skill to bring it (or it's Powers, &c.) to one Side of the Equation, and those that are known to the other side; (still keeping them to a just Equality) which the ingenious Mr *Scooten* in his *Principia Matheseos Universalis*, calls Reduction of Equations.

The Business of reducing Equations (*as of most, if not all Algebraick Operations*) is grounded and depends upon a right Application of the five *Axioms* proposed in Page 146, and therefore, if those *Axioms* be well understood, the Reason of such Operations must needs appear very plain, and the Work be easily performed; as in the following *Sections*.

Sect. 1. Of Reduction by Addition.

REDUCTION by *Addition* is grounded upon *Axiom 1.* and is only the transposing (*viz.* the removing) of any Negative Quantity from either Side of an Equation to the other Side, with the Sign $+$ before it; as in these

E X A M P L E S.

Suppose	1	$a - b = d$	Again,	1	$aa - d = c - aa$
Then	2	$a = d + b$	Let	2	$aa = c - aa + d$
For	3	$b = d$	1 + d	3	$2aa = c + d$
1 + 3	4	$a = d + b$	2 + ac		

Let	1	$3a - 4 = 6 - a$	} {	Note, When any absolute Number is registered in the Margin, you must draw a Line over it, to distinguish it from the other Numbers. As $\bar{4}$ in the 2d Step of this Example.
1 + 4	2	$3a = 6 + 4 - a$		
2 + a	3	$4a = 6 + 4 = 10$		

Let	1	$aa - dc - b = dd - 2ba$
1 + b	2	$aa - dc = dd - 2ba + b$
2 + dc	3	$aa = dd - 2ba + b + dc$
3 + 2ba	4	$aa + 2ba = dd + b + dc$

Suppose	1	$2da - d = cc - 3baa - aaa$
1 + aaa	2	$aaa + 2da - d = cc - 3baa$
2 + 3baa	3	$aaa + 3baa + 2da - d = cc$
3 + d	4	$aaa + 3baa + 2da = cc + d, \&c.$

Sect. 2. Of Reduction by Subtraction.

REDUCTION by *Subtraction* is grounded upon *Axiom 2,* and is performed by transposing (or removing) any Affirmative Quantity from either Side of the Equation, to the other Side, with the Sign $-$ before it; as in these

E X A M P L E S.

Suppose	1	$a + b - d$	Let	1	$3a + 4 = 6 + a$
And	2	$b = b$	1 - a	2	$2a + 4 = 6$
1 - 2	3	$a = d - b$	2 - 4	3	$2a = 6 - 4 = 2$

Suppose	1	$aa + dc + b = dd + 2ba$
1 - 2ba	2	$aa - 2ba + dc + b = dd$
2 - dc	3	$aa - 2ba + b = dd - dc$
3 - b	4	$aa - 2ba = dd - dc - b$

Let

Let	1	$a a a + d = c c + 3 b a a + 2 d a$
1 — 3 b a a	2	$a a a - 3 b a a + d = c c + 2 d a$
2 — 2 d a	3	$a a a - 3 b a a - 2 d a + d = c c$
3 — d	4	$a a a - 3 b a a - 2 d a = c c - d$

Sect. 3. Of Reduction by Multiplication.

FRACTIONAL Quantities, in any Equation, are brought into whole Quantities by multiplying every Term in the Equation with the Denominators of the Fractions, *per Axiom 3*; as in these

E X A M P L E S.

Suppose	1	$\frac{a}{5} = 6$
Then	2	$a = 6 \times 5 = 30.$ For $\frac{a}{5} \times 5 = \frac{5a}{5} = a.$

Let	1	$3a = \frac{dc}{2b}$	Suppose	1	$a = \frac{dd}{a-b}$
1 × 2 b	2	$6ba = dc$	1 × a — b	2	$aa - ba = dd$

Suppose	1	$\frac{aa}{b} + c + f = \frac{dx}{a}$
1 × b	2	$aa + bc + bf = \frac{dxb}{a}$
2 × a	3	$aaa + bca + bfa = dxb$

Suppose	1	$\frac{aaa}{aa-bb} = \frac{ba-bb}{a+b}$
1 × aa — bb	2	$aaa = \frac{baaa - bbba - bbba + bbbb}{a+b}$
1 × a + b	3	$aaaa + baaa = baaa - bbba - bbba + bbbb$

Sect. 4. Of Reduction by Division.

WHEN any Quantity (either known or unknown) is in every Term of an Equation, if the whole Equation be divided by that Quantity, it will be reduced into lower Terms, *per Axiom 4*, as in these following Examples.

A a 2

E X A M P L E S.

E X A M P L E S.

Suppose	1	$baa + bca = bcd$	Let	1	$aa = 7a$
$1 \div b$	2	$aa + ca = cd$	$1 \div 1a$	2	$a = 7$

Let	1	$ffaa + ffcaa - ffa = ffd a + ffd d a$
$1 \div ff$	2	$aa + caa - a = da + d d a$
$2 \div a$	3	$a + ca - 1 = d + dd$

Or when the unknown Quantity is multiplied (*viz.* joined) with any that is known; let the whole Equation be divided by the known Quantity, that so the unknown may be cleared; as in these

E X A M P L E S.

Suppose	1	$ba - ca = d$	Let	1	$caa - daa = cd - dd$
$1 \div b - c$	2	$a = \frac{d}{b - c}$	$1 \div c - d$	2	$aa = \frac{cd - dd}{c - d} = d.$

Suppose	1	$bb a a a - 2 b b a a = b d a + c b a$
$1 \div ba$	2	$baa - 2ba = d + c$
$2 \div b$	3	$aa - 2a = \frac{d + c}{b}$

Let	1	$49 d a a + 42 a a = 7 b c a + 21 c a$
$1 \div 7$	2	$7 d a a + 6 a a = b c a + 3 c a$
$2 \div a$	3	$7 d a + 6 a = b c + 3 c$
$3 \div a$	4	$a = \frac{bc + 3c}{7d + 6}$

Sect. 5. Of Reduction by Involution.

WHEN there happens to be an Equation, between any homogeneous or like Surds, take away the radical Signs from the Quantities, and they will become rational; as in these

E X A M P L E S.

Suppose	1	$\sqrt{a} = \sqrt{d + c}$	Let	1	$\sqrt[3]{a a} = \sqrt[3]{db + bc}$	<i>per Sect. 4.</i>
$1 \odot^2$	2	$a = d + c$	$1 \odot^3$	2	$aa = db + bc$	<i>Chap. 3.</i>

Or if one Side of the Equation consists of Surd Quantities, and the other Side be rational, then involve the rational Quantities to the

the same Power (or Height) with the Index of the Surd, and take away the radical Sign; as in these

E X A M P L E S.

Let	$\left \begin{array}{c c} 1 & \sqrt{a}=6 \\ \hline 1 \odot^2 & 2 \end{array} \right $	$a=36$	Suppose	$\left \begin{array}{c c} 1 & \sqrt{a}=b+c \\ \hline 1 \odot^2 & 2 \end{array} \right $	$a=bb+2bc+cc$
Suppose	$\left \begin{array}{c c} 1 & \sqrt[3]{aa-ba}=d \\ \hline 1 \odot^3 & 2 \end{array} \right $	$aa-ba=ddd$	Let	$\left \begin{array}{c c} 1 & \sqrt[5]{aa}=7 \\ \hline 1 \odot^5 & 2 \end{array} \right $	$aa=16807$

Sect. 6. Of Reduction by Evolution.

WHEN any single Power of the unknown Quantity is on one Side of an Equation; evolve both Sides of the Equation, according as the Index of that Power denotes, and their Roots will be equal; as in these

E X A M P L E S.

Suppose	$\left \begin{array}{c c} 1 & aa=36 \\ \hline 1 \cup w^2 & 2 \end{array} \right $	$a=\sqrt{36}=6$	Let	$\left \begin{array}{c c} 1 & aaa=27 \\ \hline 1 \cup w^3 & 2 \end{array} \right $	$a=\sqrt[3]{27}=3, \&c.$
Suppose	$\left \begin{array}{c c} 1 & aa=bb-dd \\ \hline 1 \cup w^2 & 2 \end{array} \right $	$a=\sqrt{bb-dd}$	Let	$\left \begin{array}{c c} 1 & aaa=b^3+3bbc+3bcc+c^3 \\ \hline 1 \cup w^3 & 2 \end{array} \right $	$a=b+c$

Or if any compound Power of the unknown Quantity be on one Side of the Equation (that hath a true Root of it's kind) evolve both Sides of the Equation, and it will be depressed into lower Terms; as in these

E X A M P L E S.

Suppose	$\left \begin{array}{c c} 1 & aa+2ba+bb=dd \\ \hline 1 \cup w^2 & 2 \end{array} \right $	$a+b=d$	$aa-2ba+bb=ddcc$	$a-b=dc$
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Here follow a few Examples of clearing Equations, wherein all the foregoing Reductions are promiscuously used, as Occasion requires.

E X A M P L E I.

Suppose	$\left \begin{array}{c c} 1 & \frac{aa+c-d}{4} = \frac{g-aa}{b}, \text{ what is } a=? \\ \hline 1 \times 4 & 2 \end{array} \right $	$aa+c-d = \frac{4g-4aa}{b}$
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$2 \times b$

$2 \times b$	3	$baa + bc - bd = 4g - 4aa$
$3 + 4aa$	4	$baa + 4aa + bc - bd = 4g$
$4 + bd$	5	$baa + 4aa + bc = 4g + bd$
$5 - bc$	6	$baa + 4aa = 4g + bd - bc$
$6 \div b + 4$	7	$aa = \frac{4g + bd - bc}{b + 4}$
$7w^2$	8	$a = \sqrt{\frac{4g + bd - bc}{b + 4}}$ as was required.

EXAMPLE 2.

Suppose	1	$\frac{a + 354}{a} = \frac{3a}{354 - a}$ what is the Value of a ?
$1 \times a$	2	$a + 354 = \frac{3aa}{354 - a}$
$2 \times 353 - a$	3	$125316 - aa = 3aa$
$2 + aa$	4	$4aa = 125316$
$4 \div 4$	5	$aa = 31329$
$5w^2$	6	$a = \sqrt{31329} = 177$, the Value of a required.

EXAMPLE 3.

Suppose	1	$\sqrt{\frac{aa + 3bb}{4}} - \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{baa}{c}} : a = ?$
$1 \odot^2$	2	$\left\{ \begin{array}{l} \frac{aa + 3bb}{4} - 2\sqrt{\frac{aa + 3bb}{4}} \times \sqrt{\frac{aa - 3bb}{4}} \\ : + \frac{aa - 3bb}{4} = \frac{baa}{c} \end{array} \right.$
That is	3	$\frac{aa}{2} - \sqrt{\frac{a^4 - 9b^4}{4}} = \frac{baa}{c}$
For		$\frac{aa + 3bb}{4} + \frac{aa - 3bb}{4} = \frac{2aa}{4} = \frac{aa}{2}$
And		$2\sqrt{\frac{aa + 3bb}{4}} = \sqrt{\frac{4aa + 12bb}{4}} = \sqrt{aa + 3bb}$
Then		$\sqrt{aa + 3bb} \times \sqrt{\frac{aa - 3bb}{4}} = \sqrt{\frac{a^4 - 9b^4}{4}}$
$3 + \sqrt{\&c.}$	4	$\frac{aa}{2} = \frac{baa}{c} + \sqrt{\frac{a^4 - 9b^4}{4}}$

4 — $\frac{b a a}{c}$	5	$\frac{a a}{2} - \frac{b a a}{c} = \sqrt{\frac{a^4 - 9 b^4}{4}}$
5 $\odot^2$	6	$\frac{a^4}{4} - \frac{b a^4}{c} + \frac{b b a^4}{c c} = \frac{a^4 - 9 b^4}{4}$
6 + $\frac{b a^4}{c}$	7	$\frac{a^4}{4} + \frac{b b a^4}{c c} = \frac{a^4 - 9 b^4}{4} + \frac{b a^4}{c}$
7 $\pm$	8	$\frac{b b a^4}{c c} + \frac{9 b^4}{4} = \frac{b a^4}{c}$
8 $\div b$	9	$\frac{b a^4}{c c} + \frac{9 b^3}{4} = \frac{a^4}{c}$
9 $\times c c$	10	$b a^4 + \frac{9 c c b^3}{4} = c a^4$
10 $\times 4$	11	$4 b a^4 + 9 c c b^3 = 4 c a^4$
11 — $4 b a^4$	12	$9 c c b^3 = 4 c a^4 - 4 b a^4$
12 $\div$	13	$a a a a = \frac{9 c c b^3}{4 c - 4 b}$
For		$4 c - 4 b \times a^4 = 4 c a^4 - 4 b a^4$
13 w^2	14	$a a = \sqrt{\frac{9 c c b^3}{4 c - 4 b}}$
14 w^2	15	$a = \sqrt{\frac{9 c c b^3}{4 c - 4 b}}$, as was required.

By Help of these Reductions (properly applied) the unknown Quantity (*a*) or it's Powers, are cleared and brought to one Side of an Equation; and if the unknown Quantity (*a*) chance to be equal to those that are known, the Question is answered: as in the first *Example* of *Seet.* 1, and 2. Or if any single Power of the unknown Quantity (*a*) is found equal to those that are known, then the respective Root of the known Quantities is the Answer; as in the first four *Examples* of *Seet.* 6, &c.

But when the Powers of the unknown Quantities are either mixed with their Root, as $a a + b a = d d$, &c; or do consist of different Powers, as $a a . a + b a a = d d$, &c: Then they are called Affected, or Adaffected Equations, which require other Methods to resolve them; viz. to find out the Value of (*a*) as shall be shewed further on.

C H A P. VI.

Of Proportional Quantities; both Arithmetical, Geometrical, and Musical.

WHAT hath been said of Numbers in *Arithmetical Progression*, Chap. 6. Part I. may be easily applied to any Series of Homogeneous or like Quantities.

Sect. I. Of Quantities in Arithmetical Progression.

THOSE Quantities are said to be in the most simple or natural Progression, that begin their Series of increase or decrease with a Cypher:

Thus $\begin{cases} 0 : a : 2a : 3a : 4a : 5a : 6a : \&c. \text{ increasing.} \\ 0 : -a : -2a : -3a : -4a : -5a : -6a : \&c. \text{ decreasing.} \end{cases}$

Or Universally, putting a the first Term in the Progression, and e the common Excess or Difference.

Then $\begin{cases} a : a + e : a + 2e : a + 3e : a + 4e : a + 5e : a + 6e : \&c. \\ a : a - e : a - 2e : a - 3e : a - 4e : a - 5e : a - 6e : \&c. \end{cases}$

In the first of these Series it is evident, that if there be but three Terms; the Sum of the Extreams will be double to the Mean.

As in these, $0 : a : 2a$: or, $a : 2a : 3a$: or, $2a : 3a : 4a$, &c. viz. $2a : + 0 = a + a$: or, $a + 3a = 2a + 2a$, &c.

Also, in the second Series, either increasing or decreasing, it is evident, that if the Terms be $a : a + e : a + 2e$, &c. increasing; then $a + a + 2e$, viz. $2a + 2e$ the Sum of the Extreams, is double to $a + e$ the Mean, or if they be $a : a - e : a - 2e$, &c. decreasing; then $a + a - 2e$: viz. $2a - 2e$, the Sum of the Extreams, is double to $a - e$ the Mean. And so it will be in any other three of the Terms. Secondly, if there are four Terms; then the Sum of the two Extreams, will be equal to the Sum of the two Means; as in these, $a : a + e : a + 2e : a + 3e$, in the Series increasing; here $a + a + 3e = a + e + a + 2e$.

Also in these, $a : a - e : a - 2e : a - 3e$, in the Series decreasing; here $a + a - 3e = a - e + a - 2e$, &c. in any other four Terms.

Consequently, If there are never so many Terms in the Series, the Sum of the two Extreams will always be equal to the Sum of

of any two Means, that are equally distant from those Extreams. As in these, $a : a + e : a + 2e : a + 3e : a + 4e : a + 5e : \&c.$ Here $a + a + 5e = a + e + a + 4e = a + 2e + a + 3e, \&c.$ And if the Number of Terms be odd, the Sum of the two Extreams will be double to the middle Term, &c. as in Corol. 1. Chap. 6. before-mentioned.

CONSECTARY 1.

Whence it follows, (and is very easy to conceive) that if the Sum of the two Extreams be multiplied into the Number of all the Terms in the Series, the Product will be double the Sum of all the Series.

Now for the easier resolving such Questions as depend upon these Progressional Quantities.

Let $\begin{cases} a = \text{the first Term, as before.} \\ y = \text{the last Term.} \\ e = \text{the common Excess, \&c. as before.} \\ N = \text{the Number of all the Terms.} \\ S = \text{the Sum of all the Series, viz. of all the Terms.} \end{cases}$

Then will $\overline{a + y} \times N = 2 S$, by the precedent Consectary : that is, $Na + Ny = 2S$. Consequently $\frac{Na + Ny}{2} = S$, the Sum of all the Series, be the Terms never so many. Thirdly, In these Series it is easy to perceive, that the common Difference (e) is so often added to the last Term of the Series ; as are the Number of Terms, except the first ; that is, the first Term (a) hath no Difference added to it, but the last Term hath so many times (e) added to it, as it is distant from the first.

Consequently, the Difference betwixt the two Extreams, is only the common Difference (e) multiplied into the Number of all the Terms less Unity or 1. That is, $\overline{N - 1} \times e = y - a$, the Difference betwixt the two Extreams, viz. $Ne - e = y - a$.

CONSECTARY 2.

Whence it follows, that if the Difference betwixt the two Extreams be divided by the Number of Terms less 1, the Quotient will be the common Difference of the Series.

To wit, $\frac{y - a}{N - 1} = e$.

B b

Now

Now by the Help of these two Confectaries, if any three of the aforesaid five Parts (*viz.* *a. y. e. N. S.*) be given; the other two may be easily found.

Thus,	1	$\frac{Na + Ny}{2} = S$	} as before.
And	2	$\frac{y - a}{N - 1} = e$	
$2 \times N - 1$	3	$y - a = Ne - e$	
$3 + e$	4	$y - a + e = Ne$	
$4 \div e$	5	$\frac{y - a + e}{e} = N$, the Number of Terms.	
1×2	6	$Na + Ny = 2S$	
$6 - Na$	7	$Ny = 2S - Na$	
$7 \div N$	8	$\frac{2S - Na}{N} = y$, the last Term.	
$6 - yN$	9	$Na = 2S - Ny$	
$9 \div N$	10	$\frac{2S - Ny}{N} = a$, the first Term.	
$6 \div a + y$	11	$\frac{2S}{a + y} = N$, the Number of Terms.	
5, and 11	12	$\frac{y - a + e}{e} = \frac{2S}{a + y}$, per Axiom 5.	
$12 \times a + y$	13	$\frac{yy - aa}{e} + a + y = 2S$	
$13 \div 2$	14	$\frac{yy - aa}{2e} + \frac{a + y}{2} = 2S$, the Sum of all the Series.	
$14 \times 2e$	15	$yy - aa + ae + ye = 2Se$	
$15 - ae$	16	$yy - aa + ye = 2Se - ae$	
$16 - ye$	17	$yy - aa = 2Se - ae - ye$	
$17 \div$	18	$\frac{yy - aa}{2S - a - y} = e$, the common Difference.	
$3 + a$	19	$Ne - e + a = y$, the last Term.	
$19 + e$	20	$Ne + a = y + e$	
$20 - Ne$	21	$y + e - Ne = a$, the first Term.	
		&c.	

In like Manner you may proceed to find out any of the five Quantities (*a. e. y. N. S.*) otherwise, *viz.* by varying or comparing those Equations one with another, you may produce new Equat ion

Equations with other Data in them; the which I shall here omit pursuing, and leave them for the Learner's Practice.

Sect. 2. Of Quantities in Geometrical Proportion.

GEOMETRICAL Proportion continued has been already defined in Sect. 2. Chap. 6. Part 1. And what is there said concerning Numbers in $\div\div$, may easily be applied to any sort of Homogeneous Quantities that are in $\div\div$.

The most natural and simple Series of Geometrical Proportionals, is when it begins with Unity or 1.

As 1 . a . aa . aaa . aaaa . a^5 . a^6 , &c. in $\div\div$

For 1 : a :: a : aa :: aa : aaa :: aaa : aaaa, &c.

Or $a . b . \frac{bb}{a} . \frac{bbb}{aa} . \frac{bbbb}{aaa} . \frac{b^5}{a^4}$, &c. are Terms in $\div\div$

For $a : b :: b : \frac{bb}{a} :: \frac{bb}{a} : \frac{bbb}{aa} :: \frac{bbb}{aa} : \frac{bbbb}{a^3} :: \frac{bbbb}{a^3} : \frac{b^5}{a^4}$, &c.

That is, when all the middle Terms betwixt the two Extrems are both Consequents and Antecedents, that Series is in Geometrical Proportion continued. Therefore in every Series of Quantities in $\div\div$ all the Terms except the last are Antecedents; and all the Terms except the first are Consequents. But universally putting a the first Term in the Series, and e the Ratio, viz. the common Multiplier, or Divisor; then it will be

$a . ae . aee . aeee . aeeee . ae^5 . ae^6$, &c. in $\div\div$

Or $a . \frac{a}{e} . \frac{a}{ee} . \frac{a}{eee} . \frac{a}{eeee} . \frac{a}{e^5}$, &c. are in $\div\div$ decreas.

For $a : ae :: ae : \frac{aee}{a} = aee$, &c.

And $a : \frac{a}{e} :: \frac{a}{e} : \frac{aa}{aee} = \frac{a}{ee} a : \frac{a}{e} :: \frac{a}{ee} : \frac{a}{eee}$, &c.

I. In any of these Series it is evident, that if three Quantities are in $\div\div$, the Rectangle of the two Extrems will be equal to the Square of the Mean; as in these, $a : ae . aee$, here $a \times aee = ae \times ae$, $= aae$, &c.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee}$; here also $a \times \frac{a}{ee} = \frac{a}{e} \times \frac{a}{e} = \frac{aa}{ee}$, &c.

II. If four Quantities are in $\div$ the Rectangle of the Extreams will be equal to the Rectangle of the Means.

As in these, $a \cdot ae \cdot aee \cdot aeee$; here $a \times ae^3 = ae \times aee$.

Or $a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee}$; here also $a \times \frac{a}{eee} = \frac{a}{e} \times \frac{a}{ee} = \frac{aa}{eee}$, &c.

Consequently, If there are never so many Terms in the Series of $\div$, the Rectangle of the Extreams will be equal to the Rectangle of any two Means that are equally distant from those Extreams.

As in these, $a \cdot ae \cdot aee \cdot aeee \cdot ae^4 \cdot ae^5$

viz. $ae^5 \times a = ae^4 \times ae$. Or $ae^5 \times a = aeee \times aee = aae^5$

III. If never so many Quantities are in $\div$ it will be, as any one of the Antecedents is to it's Consequents; so is the Sum of all the Antecedents, to the Sum of all the Consequents.

As in $\left\{ \begin{array}{l} a \cdot ae \cdot aee \cdot aeee \cdot ae^4 \cdot ae^5, \text{ \&c. increasing.} \\ a \cdot \frac{a}{e} \cdot \frac{a}{ee} \cdot \frac{a}{eee} \cdot \frac{a}{eeee} \cdot \frac{a}{e^5}, \text{ \&c. decreasing.} \end{array} \right.$

$a : ae :: a + ae + aee + ae^2 + ae^3 + ae^4 : ae + aee + ae^3 + ae^4 + ae^5$

Or $a : \frac{a}{e} :: a + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^2} + \frac{a}{e^3} : \frac{a}{e} + \frac{a}{e} + \frac{a}{ee} + \frac{a}{e^3}$

$+ \frac{a}{e} + \frac{a}{e}$ viz. $a \times \frac{ae + aee + ae^3 + ae^4 + ae^5}{ae + aee + ae^3 + ae^4} = ae$
 $\times a + ae + aee + ae^3 + ae^4.$

That is, the Rectangle of the Extreams is equal to the Rectangle of the Means; *per* Second of this *Sec.*

Note, The Ratio of any Series in $\div$ increasing, is found by dividing any of the Consequents by it's Antecedent.

Thus, $a) ae (e$ Or $ae) aee (e, \text{ \&c.}$

But if the Series be decreasing, then the Ratio is found by dividing any of the Antecedents by it's Consequent.

Thus, $\frac{a}{e}) a (e$ Or $\frac{a}{ee}) \frac{a}{e} (e, \text{ \&c.}$

CON-

CONSECTARY.

These Things being premised, such Equations may be deduced from them, as will solve all such Questions as are usually proposed about Quantities in Geometrical Proportion. In order to that,

let $\left\{ \begin{array}{l} a = \text{the first Term.} \\ e = \text{the common Ratio.} \\ y = \text{the last Term.} \\ S = \text{the Sum of all the Terms.} \end{array} \right\}$ as before.

Then $S - y =$ the Sum of all the Antecedents.

And $S - a =$ the Sum of all the Consequents.

Analogy. $1 \quad a : ae :: S - y : S - a$ per III. of this Sect.

$$1 \therefore 2 \quad Sa - aa = aeS - aey$$

$$2 \div a \quad 3 \quad S - a = eS - ey$$

$$3 + ey \quad 4 \quad S + ey - a = eS$$

$$4 - S \quad 5 \quad ey - a = eS - S$$

$$5 \div e - 1 \quad 6 \quad \frac{ye - a}{e - 1} = S, \text{ the Sum of all the Series.}$$

$$3 \div S - y \quad 7 \quad \frac{S - a}{S - y} = e, \text{ the common Ratio.}$$

$$5 + a \quad 8 \quad ey = eS + a - S$$

$$8 \div e \quad 9 \quad \frac{eS + a - S}{e} = y, \text{ the last Term.}$$

$$4 + a \quad 10 \quad S + ey = eS + a$$

$$10 - eS \quad 11 \quad S + ey - eS = a, \text{ the first Term.}$$

Note, The $\therefore$ set in the Margin at the second Step, is instead of ergo; and imports that the Rectangle of the two Extrems in the first Step, is equal to the Rectangle of the Means. And so for any other Proportion.

Sect. 3. Of Harmonical Proportion.

HARMONICAL or Musical Proportion is, when of three Quantities (or rather Numbers) the first hath the same Ratio to the third, as the Difference between the first and second, hath to the Difference between the second and third. As in these following.

Suppose a, b, c , in Musical Proportion.

$$\begin{array}{l|l|l} \text{Then} & 1 & a : c :: b - a : c - b \\ 1 \therefore & 2 & cb - ca = ac - ba \end{array}$$

$2 + ca$	3	$cb = 2ac - ba$
$3 \div 2c - b$	4	$\frac{cb}{2c - b} = a$, the first Term.
$3 + ba$	5	$2ac = cb + ba$
$5 \div c + a$	6	$\frac{2ac}{c + a} = b$, the second Term.
$5 - cb$	7	$2ac - cb = ba$
$7 \div 2a - c$	8	$\frac{ba}{2a - c} = c$, the third Term.

If there are four Terms in Musical Proportion, the first hath the same Ratio to the fourth, as the Difference between the first and second hath to the Difference between the third and fourth.

That is, let a, b, c, d , be the four Terms, &c.

Then	1	$a : d :: b - a : d - c$
1 $\therefore$	2	$db - da = da - ca$
$2 - da$	3	$db = 2da - ca$
$3 \div 2d - c$	4	$\frac{db}{2d - c} = a$
$3 \div d$	5	$b = \frac{2da - ca}{d}$
$3 + ca$	6	$db + ca = 2da$
$6 - db$	7	$ca = 2da - db$
$7 \div a$	8	$c = \frac{2da - db}{a}$
$7 \div 2a - b$	9	$\frac{ca}{2a - b} = d$.

CHAP. VII.

Of Proportion Disjunct, and how to turn Equations into Analogies, &c.

PROPORTION Disjunct, or the Rule of Three in Numbers, is already explained in *Chap. 7. Part I.* And what hath been there said, is applicable to all Homogeneous Quantities, viz. of Lines to Lines, &c.

Sect.

S E C T. I.

IF four Quantities, (*viz.* either Lines, Superficies, or Solids) be proportional: the Rectangle comprehended under the Extreams, is equal to the Rectangle comprehended under the two Means. (16 *Euclid* 6.)

For Instance, Suppose, $a . b . c . d$. to represent the four Homogeneal Quantities in Proportion, *viz.* $a : b :: c : d$; then will $ad = bc$. For suppose $b = 2a$, then will $d = 2c$, and it will be $a : 2a :: c : 2c$. Here the Ratio is 2. But $a \times 2c = 2a \times c$. *viz.* $2ca = 2ac$. Or suppose $b = 3a$ then will $d = 3c$, and it will be $a : 3a :: c : 3c$. Here the Ratio is 3. But $a \times 3c = 3a \times c$. *viz.* $3ca = 3ac$. Or universally putting e for the Ratio of the Proportion, *viz.* making $b = ae$, then will $d = ce$, and it will be $a : ae :: c : ce$. But $a \times ce = ae \times c$, *viz.* $ace = aec$. Consequently, $ad = bc$ which was to be proved.

Whence it follows, that if any three of the four proportional Quantities be given, the fourth may be easily found; thus,

Let	1	$a : b :: c : d$	
1 $\therefore$	2	$ad = bc$ as before	
$2 \div d$	3	$a = \frac{bc}{d}$	
$2 \div c$	4	$b = \frac{ad}{c}$	
$2 \div b$	5	$c = \frac{ad}{b}$	
$2 \div a$	6	$d = \frac{bc}{a}$	
$2 \div bd$	7	$\frac{a}{b} = \frac{c}{d}$	Note, In this Manner <i>Euclid</i>, in his 5th Book, expresses the Ratio of Proportionals, <i>viz.</i> the Ratio of a to b is $\frac{a}{b}$
Or $2 \div ac$	8	$\frac{b}{a} = \frac{d}{c}$	

If four Quantities are Proportionals, they will also be Proportionals in Alternation, Inversion, Composition, Division, Conversion, and Mixtly. *Euclid* 5. Def. 12, 13, 14, 15, 16.

That

That is, if	1	$a : b :: c : d$ be in direct Proportion, as before.
Then	2	$a : c :: b : d$, alternate. For $ad = bc$.
And	3	$b : a :: d : c$, inverted. For $ad = bc$.
Also	4	$a + b : b :: c + d : d$; compounded.
4 $\therefore$	5	$da + bd = bc + bd$, that is, $ad = bc$, as before.
Or	6	$a + c : c :: b + d : d$; alternately compounded.
6 $\therefore$	7	$ad + cd = bd + cd$, that is, $ad = bc$.
Again,	8	$a - b : b :: c - d : d$, divided.
8 $\therefore$	9	$ad - bd = bc - bd$, that is, $ad = bc$.
Or	10	$a - c : c :: b - d : d$, divided.
10 $\therefore$	11	$ad - cd = bc - cd$, that is, $ad = bc$.
And	12	$a : b + a :: c : d + c$, converted.
12 $\therefore$	13	$ad + ac = bc + ac$, that is, $ad = bc$.
Lastly	14	$a + b : a - b :: c + d : c - d$, mixtly.
14 $\therefore$	15	$ac - ad + bc - bd = ac + ad - bc - bd$.
15 $\pm$	16	$2bc = 2ad$, that is, $ad = bc$; as at first.

Note, What has been here done about whole Quantities in Simple Proportion, may be easily perform'd in Fractional Quantities, and Surds, &c.

For Instance, If $\frac{ab}{c} : \frac{d-c}{f} :: \frac{d+c}{c}$, and if it be required to find the fourth Term, it will be $\frac{dd-cc}{fc}$ the Rectangle of the Means; which being divided by the first Extream $\frac{ab}{c}$ will become $\frac{ab}{c} \cdot \frac{dd-cc}{fc} \left(\frac{ddc-ccc}{abfc} = \frac{dd-cc}{abf} \right)$ the fourth Term.

Or if $b : \sqrt{bd+bc} :: \sqrt{bc+bd} : \text{to a fourth Term}$. Then is, $\sqrt{bd+bc} \times \sqrt{bd+bc} = bd+bc$ the Rectangle of the Means; and $b) bd+bc (d+c$ the fourth Term. That is, $b : \sqrt{bd+bc} :: \sqrt{bd+bc} : d+c$, &c.

Sect. 2. Of Duplicate and Triplicate Proportion.

THE Proportions treated of in the last Section, are to be understood when Lines are compared to Lines, and Superficies to Superficies; or Solids to Solids, viz. when each is compared to that of it's like Kind, which is only called Simple Proportion.

But

But when Lines are compared to Superficies, or Lines are compared to Solids, such Comparisons are distinguished from the former, by the Names of Duplicate, and Triplicate, &c. Proportions; so that Simple, Duplicate, and Triplicate, &c. Proportions are to be understood in a different Sense from Simple, Double, Treble, &c. Proportions, which are only as 1, 2, 3, &c. to 1; but those of Simple, Duplicate, Triplicate, &c. Proportions, are those of $a . aa . aaa .$, &c. to 1. Or if the Simple Proportions be that of a to b , whose Ratio or Exponent is $\frac{a}{b}$ or $\frac{b}{a}$.

Then $\frac{a}{b} \times \frac{a}{b} = \frac{aa}{bb}$ is the Exponent of the Duplicate. } of $\frac{a}{b}$

And $\frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \frac{a^3}{b^3}$ is the Exponent of the Triplicate Proportions, &c. }

And if there are three, four, or more Quantities in $\div$, as $1 . a . aa . aaa . a^4 . a^5$, &c. (as in the first Series, Sect. 2. of the last Chapter.) Then, that of the first to the third, fourth, and fifth, &c. (*viz.* 1 to $aa . aaa . a^4 . a^5$) is Duplicate, Triplicate, Quadruplicate, &c. of the first to the second (*viz.* of 1 to a); and by Inversion, that of the third, fourth, fifth, is Duplicate, Triplicate, &c. of that of the second to the first (a to 1) *per Def. 10. Eucl. 5.* But the Name of these Proportions will appear more evident, and be easier understood when they are applied to Practice, and illustrated by Geometrical Figures, further on.

Sect. 3. How to turn Equations into Analogies.

FROM the first Section of this Chapter, it will be easy to conceive how to turn or dissolve Equations into Analogies or Proportions. For if the Rectangle of two (or more) Quantities, be equal to the Rectangle of two (or more) Quantities; then are those four (or more) Quantities Proportional. By the 16 *Eucl. 6.* That is, if $ab = cd$, then is $a : c :: d : b$, or $c : a :: b : d$, &c. From whence there arises this general Rule for turning Equations into Analogies.

C c

R U L E.

R U L E.

Divide either Side of the given Equation (if it can be done) into two such Parts, or Factors, as being multiplied together will produce that Side again; and make those two Parts the two Extreams. Then divide the other Side of the Equation (if it can be done) in the same Manner as the first was, and let those two Parts or Factors be the two Means.

For Instance, Suppose $ab + ad = bd$. Then $a : b :: d : b + d$, or $b : a :: b + d : d$, &c. Or taking ad from both Sides of the Equation, and it will be $ab = bd - ad$; then $a : d :: b - a : b$, or, $b : d :: b - a : a$, &c.

Again, suppose $aa + 2ae = 2by + yy$. Here a and $a + 2e$ are the two Factors of the first Side in this Equation; for $a + 2e \times a = aa + 2ae$.

Again, y and $2b + y$ are the two Factors of the other Side; therefore, $a : y :: 2b + y : a + 2e$, or $2b + y : a + 2e :: a : y$, &c.

When one Side of any Equation can be divided into two Factors, as before, and the other Side cannot be so divided, then make the Square Root of that Side either the two Extreams or the two Means. For Instance, Suppose $bc + bd = da + g$, then $b : \sqrt{da + g} :: \sqrt{da + g} : c + d$, or $\sqrt{da + g} : b :: c + d : \sqrt{da + g}$, &c.

C H A P. VIII.

Of Substitution, and the Solution of Quadratick Equations.

Sect. I. Of Substitution.

WHEN new Quantities not concerned in the first stating of any Question, are put instead of some that are engaged in it, that is called *Substitution*. For Instance, If instead of $\sqrt{bc - dc}$ you put z , or any other Letter; that is, make $z = \sqrt{bc - dc}$. Or suppose $aa + ba - ca + da = dc$, instead of $b - c + d$ put s , or any other Letter not engaged with the Question, viz. $s = b - c + d$, then $aa + sa = dc$. That is, if c be greater than

than $b + d$, it is $aa - sa = dc$; but if $b + d$ be greater than c , then it is $aa + sa = dc$.

And this way of substituting or putting of new Quantities instead of others, may be found very useful upon several Occasions; viz. in Order to make some following Operations in the Question more easy, and perhaps much shorter than they would be without it, as you may observe in some Questions hereafter proposed in this Tract.

And when those Operations, in which the substituted Quantities were assisting or useful, are performed according as the Nature of the Question required, you may then (if there be Occasion) bring the original or first Quantities into the Equation, in the Place (or Places) of those substituted Quantities, which is called Restitution, as you may see further on.

Sect. 2. The Solution of Quadratick Equations.

WHEN the Quantity sought is brought to an Equality with those that are known, and is on one Side of the Equation, in no more than two different Powers whose Indices are double one to another, those Equations are called Quadratick Equations Adfectèd; and do fall under the Consideration of three Forms or Cases.

$$\begin{array}{l} \text{Case 1. } aa + 2ba = dc. \\ \text{Case 2. } aa - 2ba = dc. \\ \text{Case 3. } 2ba - aa = dc. \end{array} \left. \vphantom{\begin{array}{l} \text{Case 1.} \\ \text{Case 2.} \\ \text{Case 3.} \end{array}} \right\} \text{ And } \begin{cases} a^4 + 2ba^2 = dc. \\ a^4 - 2ba^2 = dc. \\ 2ba^2 - a^4 = dc. \end{cases}$$

$$\text{Also } \begin{cases} a^6 + 2ba^3 = dc. \\ a^6 - 2ba^3 = dc. \\ 2ba^3 - a^6 = dc. \end{cases} \left. \vphantom{\begin{cases} a^6 + 2ba^3 = dc. \\ a^6 - 2ba^3 = dc. \\ 2ba^3 - a^6 = dc. \end{cases}} \right\} \text{ And } \begin{cases} a^8 + 2ba^4 = dc. \\ a^8 - 2ba^4 = dc. \\ 2ba^4 - a^8 = dc. \end{cases} \text{ \&c.}$$

When there happens to be more Terms in one of these Kind of Equations than two, and the highest Power of the unknown Quantity is multiplied into some known Co-efficients; you must reduce them by *Division*; as in Sect. 4. of Chap. 5. and for the Fractional Quantities that may arise by those Divisions, substitute another Quantity doubled.

For Instance, let $baa + caa - ca - da = dc + cb$, then $aa - \frac{ca - da}{b + c} = \frac{dc + cb}{b + c}$. Make $\frac{c - d}{b + c} = 2x$, and if you please,

for $\frac{dc + cb}{b + c}$ put z . Then will $aa - 2xa = z$ be the new Equation, equal to the other, being now fitted for a Solution.

Now any of these three Forms of Equations being thus prepared for a Solution, may be reduced to simple Powers by casting off the second or lowest Term of the unknown Quantity; which is done by Substitution; thus, always take half the known Co-efficient, and add it to (Case 1.) or subtract it from (Case 2.) it's fellow Factor; and for their Sum, or Difference, Substitute another Letter; as in these.

Let	1	$aa + 2ba = dc$ Case 1.
Put	2	$a + b = e$
$2 \odot^2$	3	$aa + 2ba + bb = ee$
$3 - 1$	4	$bb = ee - dc$
$4 + dc$	5	$ee = bb + dc$
$5 \omega^2$	6	$e = \sqrt{bb + dc}$
2 and 6	7	$a + b = \sqrt{bb + dc}$, per Axiom 5.
$7 - b$	8	$a = \sqrt{bb + dc} - b$

Again,		
Let	1	$aa - 2ba = dc$ Case 2.
Put	2	$a - b = e$
$2 \odot^2$	3	$aa - 2ba + bb = ee$
$3 - 1$	4	$bb = ee - dc$
$4 + dc$	5	$ee = dc + bb$
$5 \omega^2$	6	$e = \sqrt{dc + bb}$
2 and 6	7	$a - b = \sqrt{dc + bb}$
$7 + b$	8	$a = b + \sqrt{dc + bb}$

In Case 3. From Half the known Co-efficient subtract it's fellow Factor.

Thus, Let	1	$2ba - aa = dc$
Put	2	$b - a = e$
$2 \odot^2$	3	$bb - 2ba + aa = ee$
$1 + 3$	4	$bb = dc + ee$
$4 - dc$	5	$ee = bb - dc$

$5 \omega^2$

5 uv^2	6	$e = \sqrt{bb - dc}$
2 and 6	7	$b - a = \sqrt{bb - dc}$
7 + a	8	$b = a + \sqrt{bb - dc}$
8 - $\sqrt{}$, &c.	9	$a = b - \sqrt{bb - dc}$

And this Method holds good in those other Equations, wherein the highest Powers are a^4 , a^6 , a^8 , &c. As, for instance,

Let	1	$a^6 + 2b^3 = dc$ Case 1.
Put	2	$a^3 + b = e$
2 $\odot^2$	3	$a^6 + 2ba^3 + bb = ee$
3 - 1	4	$bb = ee - dc$
4 + cd	5	$ee = bb + dc$
5 uv^2	6	$e = \sqrt{bb + dc}$
2 and 6	7	$a^3 + b = \sqrt{bb + dc}$
7 - b	8	$a^3 = \sqrt{bb + dc} - b$
8 uv^3	9	$a = \sqrt[3]{\sqrt{bb + dc} - b}$

The same may be done with all the rest, Care being taken to add, or subtract, according as the Case requires.

But all Quadratick Equations may be more easily resolved by completing the Square, which is grounded upon the Consideration of raising a Square from any Binomial, or Residual Root. (See Sect. 5. Chap. 1.) *Viz.* if $a + b$ be involved to a Square, it will be $aa + 2ba + bb$; and if $a - b$ be so involved, it will be $aa - 2ba + bb$. Whence it is easy to observe, that $aa + 2ba = dc$ (Case 1.), and $aa - 2ba = dc$ (Case 2.), are imperfect Squares, wanting only bb to make them compleat. And therefore it is, that if half the known Co-efficient be involved to the second Power, and the Square be added to both Sides of the Equation, the unknown Side will become a compleat Square.

Thus Let	1	$aa + 2ba = dc$	Here half the Co-efficient $2b$ is b , which being squared, is bb .
But	2	$bb = bb$	
1 + 2	3	$aa + 2ba + bb = dc + bb$ Case 1.	
3 uv^2	4	$a + b = \sqrt{dc + bb}$, as before.	

Again.

Again.

Let	1	$aa - 2ba = dc$	Case 2.
But	2	$bb = bb$	
1 + 3	3	$aa - 2ba + bb = dc + bb$	
3 w^2	4	$a - b = \sqrt{dc + bb}$, &c. as before.	

But in Case 3. you must change the Signs of all the Terms in the Equation,

Thus	1	$2ba - aa = dc$	Case 3.
1 +	2	$aa - 2ba = -dc$	
Then	3	$aa - 2ba + bb = bb - dc$, &c.	

And this Method of completing the Square, holds true in those other Equations.

Viz.	1	$aaaa + 2baa = dc$	Case 1.
For	2	$bb = bb$, as before.	
1 + 2	3	$aaaa + 2baa + bb = dc + bb$	
3 w^2	4	$aa + b = \sqrt{dc + bb}$	
4 - b	5	$aa = \sqrt{dc + bb} : -b$	
5 w^2	6	$a = \sqrt{\sqrt{dc + bb} : -b}$, and so on for the rest.	

Or let	1	$a^6 + 2baaa = dc$, as before, Case 1.
And	2	$bb = bb$
1 + 2	3	$a^6 + 2baaa + bb = dc + bb$
1 w^2	4	$aaa + b = \sqrt{dc + bb}$
4 - b	5	$aaa = \sqrt{dc + bb} : -b$
5 w^3	6	$a = \sqrt[3]{\sqrt{dc + bb} : -b}$, &c.

COROLLARY.

Hence it is evident, that whatsoever Method is used in solving these (or indeed any other) Equations, the Result will still be the same, if the Work be true; as you may observe from the Operations of this Section: for both these Methods here proposed, give the same Theorems in their respective Cases for the Value of (a).

Thus

Thus, when $aa + 2ba = dc$, then

Theorem 1. $a = \sqrt{dc + bb} : -b$

And when $aa - 2ba = dc$, then

Theorem 2. $a = b + \sqrt{dc + bb}$

Again, when $2ba - aa = dc$, then

Theorem 3. $a = b - \sqrt{bb - dc}$

The like *Theorems* may be easily raised for the rest.

If the known Co-efficients (of the second or lowest Term) be any single Quantity, as $aa + ba = dc$, &c. then is $\frac{1}{2}b$ it's Half, and $\frac{1}{4}bb$ will be the Square of that Half; that is, $\frac{1}{2}b \times \frac{1}{2}b = \frac{1}{4}bb$, and then the Work will stand

Thus	1	$aa + ba = dc$
1 C $\square$	2	$aa + ba + \frac{1}{4}bb = dc + \frac{1}{4}bb$
2 uv^2	3	$a + \frac{1}{2}b = \sqrt{dc + \frac{1}{4}bb}$
3 $-\frac{1}{2}b$	4	$a = \sqrt{dc + \frac{1}{4}bb} : -\frac{1}{2}b$, and so for the rest.

Note, C $\square$ placed in the Margin against the second Step, signifies that the imperfect Square $aa + ba$ in the first Step, is there compleated, viz. in the second Step.

Now by the help of these *Theorems*, it will be easy to calculate or find the Value of the unknown Quantity (a) in Numbers.

E X A M P L E 1.

Suppose $aa + 2ba = z$. Let $b = 16$, and $z = 4644$.

then $a = \sqrt{z + bb} : -b$ per *Theorem 1.*

But $z + bb = 4644 + 256 = 4900$, and $\sqrt{4900} = 70$

Consequently $a = 70 - 16$. viz. $a = 54$.

But every Affected Equation, hath as many Roots (or rather Values of the unknown Quantity), either real or imaginary, as are the Dimensions (viz. the Index) of it's highest Power; and therefore the Quantity a , in this Equation, hath another Value either Affirmative or Negative; which may be thus found.

The given Equation is $aa + 32a = 4644$, and it's Root $a = 54$.

Let these two Equations be made equal or equated to 0, viz. to Nothing.

Thus,

Thus, $aa + 32a - 4644 = 0$, and $a - 54 = 0$.

Then divide the given Equation by it's first Root, and the Quotient will shew the second Value of a .

$$\begin{array}{r} \text{Thus, } a - 54 = 0) \quad aa + 32a - 4644 = 0 \quad (a + 86 = 0 \\ \quad \quad \quad aa - 54a \\ \hline \quad \quad \quad + 86a - 4644 \\ \quad \quad \quad 86a - 4644 \\ \hline \quad \quad \quad (0) \end{array}$$

Hence the second Value of a is $= -86$, or $86 = -a$, which seems impossible, viz. that an Affirmative Quantity should be equal to a Negative Quantity; yet even by this second Value of a , and the same Co-efficient, the true (or first) Equation may be formed

$$\begin{array}{l|l} \text{Thus, Let } 1 & a = -86 \\ 1 \text{ } \odot^2 & 2 \quad aa = +7396, \text{ viz. } -86 \times -86 = +7396 \\ 1 \times 32 & 3 \quad 32a = -2752 \\ 2 + 3 & 4 \quad aa + 32a = 4644, \text{ as at first.} \end{array}$$

EXAMPLE 2.

$$\begin{array}{l|l} \text{Suppose } 1 & aa - 7a = 948,75, \text{ then per Theorem 2.} \\ 1 \text{ } \odot \square & 2 \quad aa - 7a + \frac{49}{4} = 948,75 + \frac{49}{4} = 961 \\ 2 \text{ } \underline{uw^2} & 3 \quad a - \frac{7}{2} \text{ (or } 3,5) = \sqrt{961} = 31 \\ 3 + 3,5 & 4 \quad a = 31 + 3,5 = 34,5 \end{array}$$

Again, for the second Value of a , let $aa - 7a - 948,75 = 0$, and $a - 34,5 = 0$. Then

$$a - 34,5 = 0) \quad aa - 7a - 948,75 = 0 \quad (a + 27,5 = 0.$$

Consequently this second Value is $a = -27,5$ which will form the original Equation, $aa - 7a = 948,75$ if it be ordered as the last was.

EXAMPLE 3.

Suppose $36a - aa = 243$, then per Theorem 3. $a = 18 - \sqrt{324 - 243}$, viz. half 36 squared is 324, &c. that is, $a = 18 - \sqrt{81}$; but $\sqrt{81} = 9$, therefore $a = 18 - 9 = 9$. Now this third Form is called an ambiguous Equation, because it hath two Affirmative Values of the unknown Quantity (a), both which may be found without such Division as was used before.

before. For in this Case, $a = 18 + \sqrt{81}$, viz. $a = 18 + 9 = 27$, or, $a = 18 - 9 = 9$, as before. And both these Values of a are equally true, as to forming the given Equation; viz. $36a - aa = 243$. For if $a = 9$, then $aa = 81$, and $36a = 324$; but $324 - 81 = 243$, therefore $a = 9$.

Again, if $a = 27$, then will $aa = 729$, and $36a = 972$: But $972 - 729 = 243$; consequently it may be, $a = 27$. Now either of these Values of a may be found by *Division*, as those were in the other two Cases, one of them being first found by the Theorem. Thus, let $36a - aa - 243 = 0$, and $9 - a = 0$, then $9 - a = 0$ $36a - aa - 243 = 0$ ($a - 27 = 0$)

$$\begin{array}{r} 9a - aa \\ 27a - 0 = 243 \\ 27 \quad \quad 243 \\ \hline (0) \quad \quad (0) \end{array}$$

Hence, if $a - 27 = 0$, then $a = 27$, as before.

Notwithstanding all Quadratick Equations of this third Form have two Affirmative Roots (as in this), yet but one of those Roots will give a true Answer to the Question, and that is to be chosen according to the Nature and Limits of the Question, as shall be shewed further on.

SCHOLIUM.

From the Work of the three last Examples, it may be observed; that the Sum of both the Roots will always be equal to the Co-efficient of their respective Equations, with a contrary Sign.

Thus. In Example 1. $aa + 32a = 4644$

$$\begin{array}{r} \text{Here } a = 54 \\ \text{And } a = -86 \end{array} \left. \vphantom{\begin{array}{r} \text{Here } a = 54 \\ \text{And } a = -86 \end{array}} \right\} \text{Add}$$

$$2a = -32$$

In Example 2.

$$\begin{array}{r} aa - 7a = 948,75 \\ \text{Here } a = 34,5 \\ \text{And } a = -27,5 \end{array} \left. \vphantom{\begin{array}{r} \text{Here } a = 34,5 \\ \text{And } a = -27,5 \end{array}} \right\} \text{Add}$$

$$2a = +7$$

In the last Example $36a - aa = 243$
Which was changed into $aa - 36a = -243$

$$\begin{array}{r} \text{Here } a = 9 \\ \text{And } a = 27 \end{array} \left. \vphantom{\begin{array}{r} \text{Here } a = 9 \\ \text{And } a = 27 \end{array}} \right\} \text{Add}$$

$$2a = 36$$

D d

Hence

Hence it is evident, that if either of the Roots be found, the other may be easily had without Divisions.

If the Contents of this Section be well understood, it will be easy to give a Numerical Solution to any Quadratick Equation, that happens to arise in resolving of Questions, &c. And as for giving a Geometrical Construction of them, I think it not proper in this Place; because I here suppose the Learner wholly ignorant of the first Principles of Geometry, therefore I shall refer that Work to the next Part.

C H A P. IX.

Of Analysis, or the Method of resolving Problems exemplified by Variety of Numerical Questions.

N.B. **H**ERE I advise the Learner to make use always of the same Letters, to represent the same Data in all Questions.

Viz. { If a represent any Number } or other Quantity,
 { And e represent a less Number }

Then let { $a + e = s$ their Sum.
 $a - e = d$ their Difference.
 $ae = p$ their Product.
 $\frac{a}{e} = q$ their Quotient.
 $aa + ee = z$ the Sum of their Squares.
 $aa - ee = x$ the Difference of their Squares.

Any two of these six (s, d, p, q, z, x) being given, thence to find the rest; which admits of fifteen Variations, or Questions.

Question 1. Suppose s and d were given, and it were required by them to find $a . e . p . q . z .$ and x .

Let { $\begin{array}{l|l} 1 & a + e = s \\ 2 & a - e = d \end{array}$ } and suppose { $\begin{array}{l} s = 240 \\ d = 192 \end{array}$ } Then
 $1 + 2$ 3 $2a = s + d = 432$
 $3 \div 2$ 4 $a = \frac{s+d}{2} = 216$, here a is found.
 $1 - 2$ 5 $2e = s - d = 48$.

$5 \div 2$

$5 \div 2$	6	$e = \frac{s-d}{2} = 24$, here e is found.
4×6	7	$ae = \frac{ss-dd}{4} = p = 5184$, here p is found.
$4 \div 6$	8	$\frac{a}{e} = \frac{s+d}{s-d} = q = 9$, here q is found.
$4 \odot^2$	9	$aa = \frac{ss + 2sd + dd}{4} = 46656$
$6 \odot^2$	10	$ee = \frac{ss - 2sd + dd}{4} = 576$
$9 + 10$	11	$aa + ee = \frac{ss + dd}{2} = z = 47232$, z found.
$9 - 10$	12	$aa - ee = sd = x = 46080$, x found.

Question 2. Let s and p be given, to find the rest.

That is {		1	$a + e = s = 240$	} Quære $a . e . d . q . z . x$.
		2	$ae = p = 5184$	
$1 \odot^2$	3	$aa + 2ae + ee = ss =$	57600	
2×4	4	$4ae = 4p =$	20736	
$3 - 4$	5	$aa - 2ae + ee = ss - 4p =$	36864	
$5 \omega^2$	6	$a - e = \sqrt{ss - 4p} = d =$	192	
$1 + 6$	7	$2a = s + \sqrt{ss - 4p}$		
$7 \div 2$	8	$a = \frac{s + \sqrt{ss - 4p}}{2}$, hence $a =$	216	
$1 - 6$	9	$2e = s - \sqrt{ss - 4p}$		
$9 \div 2$	10	$e = \frac{s - \sqrt{ss - 4p}}{2}$, hence $e =$	24.	
$8 \div 10$	11	$\frac{a}{e} = \frac{s + \sqrt{ss - 4p}}{s - \sqrt{ss - 4p}} = q =$	9.	
$8 \odot^2$	12	$aa = \frac{ss + s\sqrt{ss - 4p}}{2} : - p$		
$10 \odot^2$	13	$ee = \frac{ss - s\sqrt{ss - 4p}}{2} : - p$		
$12 + 13$	14	$aa + ee = ss - 2p = z =$	47232	
$12 - 13$	15	$aa - ee = s\sqrt{ss - 4p} = x =$	46080	

Question 3. Suppose s and q are given, to find the rest.

<i>Viz.</i>	1	$a + e = s. = 240$	} Quære $a.e.d.p.z.x,$
	2	$\frac{a}{e} = q = 9$	
$2 \times e$	3	$a = qe$	
$1 - 3$	4	$e = s - qe$	
$4 + qe$	5	$qe + e = s$	
$5 \div q + 1$	6	$e = \frac{s}{q+1},$ for $q+1 \times e = qe + e$	
$1 - 6$	7	$a = s - \frac{s}{q+1} = \frac{qs}{q+1}$	
6×7	8	$ae = \frac{qss}{qq + 2q + 1} = p$	
$7 - 6$	9	$a - e = \frac{qs - s}{q+1} = d$	
$7 \odot^2$	10	$aa = \frac{qqss}{qq + 2q + 1}$	
$6 \odot^2$	11	$ee = \frac{ss}{qq + 2q + 1}$	
$10 + 11$	12	$aa + ee = \frac{qqss + ss}{qq + 2q + 1} = z$	
$10 - 11$	13	$aa - ee = \frac{qqss - ss}{qq + 2q + 1} = x$	

Question 4. Let s and z be given, to find the rest.

<i>Viz.</i>	1	$a + e = s. = 240$	} Quære $a.e.d.p.q.x,$
	2	$aa + ee = z. = 47232$	
$1 \odot^3$	3	$aa + 2ae + ee = ss$	
$3 - 2$	4	$2ae = ss - z$	
$2 - 4$	5	$aa - 2ae + ee = 2z - ss$	
$5 \sqrt{w}^3$	6	$a - e = \sqrt{2z - ss} = d$	

$$\begin{array}{l|l|l}
 1 + 6 & 7 & 2a = s + \sqrt{2z - ss} \\
 7 \div 2 & 8 & a = \frac{s + \sqrt{2z - ss}}{2} \\
 1 - 6 & 9 & 2e = s - \sqrt{2z - ss} \\
 9 \div 2 & 10 & e = \frac{s - \sqrt{2z - ss}}{2}
 \end{array}$$

The rest are found just as in the 2d *Question*; the 8 and 10 Steps here being the very same with the 8 and 10 Steps there.

Question 5. When s and x are given, to find the rest.

$$\begin{array}{l|l|l}
 \text{Viz. } \{ & 1 & a + e = s = 240 \\
 & 2 & aa - ee = x = 46080 \} \text{ Quære } a.e.d.p.q.z. \\
 2 \div 1 & 3 & a - e = \frac{x}{s} = d, \text{ viz. } a + e) aa - ee (a - e \\
 1 + 3 & 4 & 2a = s + \frac{x}{s} = \frac{ss + x}{s} \\
 4 \div 2 & 5 & a = \frac{ss + x}{2s} \\
 1 - 3 & 6 & 2e = s - \frac{x}{s} = \frac{ss - x}{s} \\
 6 \div 2 & 7 & e = \frac{ss - x}{2s} \\
 5 \times 7 & 8 & ae = \frac{ssss - xx}{4ss} = p \\
 5 \div 7 & 9 & \frac{a}{e} = \frac{ss + x}{ss - x} = q \\
 5 \odot^2 & 10 & aa = \frac{s^4 + 2ssx + xx}{4ss} \\
 7 \odot^2 & 11 & ee = \frac{s^4 - 2ssx + xx}{4ss} \\
 10 + 11 & 12 & aa + ee = \frac{s^4 + xx}{2ss} = z
 \end{array}$$

Question

Question 6. Suppose d and p are given, to find the rest.

Viz. {	1	$a - e = d = 192$	} Quære $a.e.s.q.z.x.$
	2	$ae = p = 5184$	
$1 \odot^2$	3	$aa - 2ae + ee = dd$	
2×4	4	$4ae = 4p$	
$3 + 4$	5	$aa + 2ae + ee = dd + 4p$	
$5 w^2$	6	$a + e = \sqrt{dd + 4p} = s$	
$6 + 1$	7	$2a = d + \sqrt{dd + 4p}$	
$7 \div 2$	8	$a = \frac{d + \sqrt{dd + 4p}}{2}$	
$6 - 1$	9	$2e = \sqrt{dd + 4p} \sqrt{-d}$	
$9 \div 2$	10	$e = \frac{\sqrt{dd + 4p} \sqrt{-d}}{2}$	
$8 \div 10$	11	$\frac{a}{e} = \frac{d + \sqrt{dd + 4p}}{\sqrt{dd + 4p} \sqrt{-d}} = q$	
$8 \odot^2$	12	$aa = \frac{dd + 2p + d\sqrt{dd + 4p}}{2}$	
$10 \odot^2$	13	$ee = \frac{dd + 2p - d\sqrt{dd + 4p}}{2}$	
$12 + 13$	14	$aa + ee = dd + 2p = z$	
$12 - 13$	15	$aa - ee = d\sqrt{dd + 4p} = x$	

Question 7. Let d and q be given, to find the rest.

Viz. {	1	$a - e = d = 192$	} Quære $a.e.s.p.z.x.$
	2	$\frac{a}{e} = q = 9$	
$2 \times e$	3	$a = qe$	
$1 + e$	4	$a = d + e$	
3 and 4	5	$qe = d + e$	
$5 - e$	6	$qe - e = d$	
$6 \div q - 1$	7	$e = \frac{d}{q - 1}, \text{ for } q - 1 \times e = qe - e$	
$1 + 7$	8	$a = d + \frac{d}{q - 1} = \frac{qd}{q - 1}$	
$7 + 8$	9	$a + e = \frac{qd + d}{q - 1} = s$	

7 x 8

7×8	10	$ae = \frac{qdd}{qq - 2q + 1} = p$
$8 \odot^2$	11	$aa = \frac{qqdd}{qq - 2q + 1}$
$7 \odot^2$	12	$ee = \frac{dd}{qq - 2q + 1}$
$11 + 12$	13	$aa + ee = \frac{qqdd + dd}{qq - 2q + 1} = x$
$11 - 12$	14	$aa - ee = \frac{qqdd - dd}{qq - 2q + 1} = x$

Question 8. Suppose d and z given, to find the rest.

Viz. {	1	$a - e = d = 192$	} Quære $a.e.s.p.q.x.$
	2	$aa + ee = z = 47232$	
$1 \odot^2$	3	$aa - 2ae + ee = dd$	
$2 - 3$	4	$2ae = z - dd$	
$2 + 4$	5	$aa + 2ae + ae = 2z - dd$	
$5 \omega^2$	6	$a + e = \sqrt{2z - dd} = s$	
$1 + 6$	7	$2a = d + \sqrt{2z - dd}$	
$7 \div 2$	8	$a = \frac{d + \sqrt{2z - dd}}{2}$	
$6 - 1$	9	$2e = \sqrt{2z - dd} - d$	
$9 \div 2$	10	$e = \frac{\sqrt{2z - dd} - d}{2}$	
8×10	11	$ae = \frac{z - dd}{2} = p$	
$8 \odot^2$	12	$aa = \frac{z + d\sqrt{2z - dd}}{2}$	
$10 \odot^2$	13	$ee = \frac{z - d\sqrt{2z - dd}}{2}$	
$12 - 13$	14	$aa - ee = d\sqrt{2z - dd} = x$	
$8 \div 10$	15	$\frac{a}{e} = \frac{d + \sqrt{2z - dd}}{\sqrt{2z - dd} - d} = q$	

Question

Question 9. Let d and x be given, to find the rest.

Viz. {	1	$a - e = d = 240$	} Quære $a.e.s.p.q.z.$
	2	$aa - ee = x = 46080$	
$2 \div 1$	3	$a + e = \frac{x}{d} = s$, viz. $a - e$	$aa - ee$
$1 + 3$	4	$2a = \frac{dd + x}{d}$	
$4 \div 2$	5	$a = \frac{dd + x}{2d}$	
$3 - 5$	6	$e = \frac{x - dd}{2d}$	
5×6	7	$ae = \frac{xx - d^4}{4dd} = p$	
$5 \div 6$	8	$\frac{a}{e} = \frac{dd + x}{x - dd} = q$	
$5 \odot^2$	9	$aa = \frac{d^4 + 2ddx + xx}{4dd}$	
$6 \odot^2$	10	$ee = \frac{xx - 2ddx + d^4}{4dd}$	
$9 + 10$	11	$aa + ee = \frac{d^4 + xx}{2dd} = z$	

Question 10. Let p and q be given, to find the rest.

Viz. {	1	$ae = p = 5184$	} Quære $a.e.d.z.x.$
	2	$\frac{a}{e} = q = 9$	
1×2	3	$aa = qp$, for $\frac{ae}{1} \times \frac{a}{e} = \frac{aae}{e} = aa$	
$3 \omega^2$	4	$a = \sqrt{qp}$	
$1 \div 2$	5	$ee = \frac{p}{q}$, for $\frac{a}{1} \frac{ae}{1} \left(\frac{ae}{a} = ee \right)$	
$5 \omega^2$	6	$e = \sqrt{\frac{p}{q}}$	
$4 + 6$	7	$a + e = \sqrt{qp} + \sqrt{\frac{p}{q}} = s$	

4 - 6	8	$a - e = \sqrt{qp} - \sqrt{\frac{p}{q}} = d$
3 + 5	9	$aa + ee = qp + \frac{p}{q} = z$
3 - 5	10	$aa - ee = pq - \frac{p}{q} = x$

Question 11. Let p and z be given, to find the rest.

<i>Viz.</i> {	1	$ae = p = 5184$	} Quære $a . e . \&c.$
	2	$aa + ee = z = 47232$	
1×2	3	$2ae = 2p$	
$2 + 3$	4	$aa + 2ae + ee = z + 2p$	
$4 \sqrt{w^2}$	5	$a + e = \sqrt{z + 2p} = s$	
$2 - 3$	6	$aa - 2ae + ee = z - 2p$	
$6 \sqrt{w^2}$	7	$a - e = \sqrt{z - 2p} = d$	
$5 + 7$	8	$2a = \sqrt{z + 2p} + \sqrt{z - 2p}$	
$8 \div 2$	9	$a = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{2}$	
$5 - 7$	10	$2e = \sqrt{z + 2p} - \sqrt{z - 2p}$	
$10 \div 2$	11	$e = \frac{\sqrt{z + 2p} - \sqrt{z - 2p}}{2}$	
$9 \div 11$	12	$\frac{a}{e} = \frac{\sqrt{z + 2p} + \sqrt{z - 2p}}{\sqrt{z + 2p} - \sqrt{z - 2p}} = q$	
$9 \odot^2$	13	$aa = \frac{z + \sqrt{zz - 4pp}}{2}$	
$11 \odot^2$	14	$ee = \frac{z - \sqrt{zz - 4pp}}{2}$	
$aa - ee$	15	$aa - ee = \sqrt{zz - 4pp} = x$	

Question 12. Let p and x be given, to find the rest.

<i>Viz.</i> {	1	$ae = p = 5184$	} Quære $a . e . \&c.$
	2	$aa - ee = x = 46080$	
$1 \odot^2$	3	$aaee = pp$	

E e

2 $\odot^2$

$2 \text{ @ } 2$	4	$aaaa - 2aaee + eeee = xx$
3×4	5	$4aaee = 4pp$
$4 + 5$	6	$aaaa + 2aaee + eeee = xx + 4pp$
$6 \omega^2$	7	$aa + ee = \sqrt{xx + 4pp} = z$
$2 + 7$	8	$2aa = x + \sqrt{xx + 4pp}$
$8 \div 2$	9	$aa = \frac{x + \sqrt{xx + 4pp}}{2}$
$9 \omega^2$	10	$a = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}}$
$7 - 2$	11	$2ee = \sqrt{xx + 4pp} - x$
$11 \div 2$	12	$ee = \frac{\sqrt{xx + 4pp} - x}{2}$
$12 \omega^2$	13	$e = \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}}$
$10 + 13$	14	$a + e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} + \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}} = s$
$10 - 13$	15	$a - e = \sqrt{\frac{x + \sqrt{xx + 4pp}}{2}} - \sqrt{\frac{\sqrt{xx + 4pp} - x}{2}} = d$
$9 + 12$	16	$aa + ee = \sqrt{xx + 4pp} = z$

Question 13. Having q and z given, to find the rest.

$\text{Viz. } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right.$	$\left\{ \begin{array}{l} \frac{a}{e} = q = 9 \\ aa + ee = z = 47232 \end{array} \right.$	$\left. \begin{array}{l} \\ \end{array} \right\} \text{Quære } a, e, \&c.$
$1 \times e$	3	$a = qe$
$3 \text{ @ } 2$	4	$aa = qqee$
$2 - 4$	5	$ee = z - qqee$
$4 + qqee$	6	$qqee + ee = z$
$6 \div qq + 1$	7	$ee = \frac{z}{qq + 1}, \text{ for } qq + 1 \times ee = qqee + ee$

2 - 7	8	$aa = z - \frac{z}{qq + 1} = \frac{qqz}{qq + 1}$
8 uv^2	9	$a = \sqrt{\frac{qqz}{qq + 1}}$
7 uv^2	10	$e = \sqrt{\frac{z}{qq + 1}}$
9 + 10	11	$a + e = \sqrt{\frac{qqz}{qq + 1}} + \sqrt{\frac{z}{qq + 1}} = s$
9 - 10	12	$a - e = \sqrt{\frac{qqz}{qq + 1}} - \sqrt{\frac{z}{qq + 1}} = d$
9 x 10	13	$ae = \sqrt{\frac{qqzz}{q^2 + 2qq + 1}} = p$
8 - 7	14	$aa - ee = \frac{qqz - z}{qq + 1} = x$

Question 14. When q and x are given, to find the rest.

Viz. {	1	$\frac{a}{e} = q = 9$	} Quære $a . e . \&c.$
	2	$aa - ee = x = 46080$	
1 x e	3	$a = qe$	
3 $\&^2$	4	$aa = qqee$	
2 + ee	5	$aa = x + ee$	
4 and 5	6	$qqee = x + ee$	
6 - ee	7	$qqee - ee = x$	
7 ÷ qq - 1	8	$ee = \frac{x}{qq - 1}$	
2 + 8	9	$aa = x + \frac{x}{qq - 1} = \frac{qqx}{qq - 1}$	
9 uv^2	10	$a = \sqrt{\frac{qqx}{qq - 1}}$	
8 uv^2	11	$e = \sqrt{\frac{x}{qq - 1}}$	
10 + 11	12	$a + e = \sqrt{\frac{qqx}{qq - 1}} + \sqrt{\frac{x}{qq - 1}} = s$	

10 - 11	13	$a - e = \sqrt{\frac{qqx}{qq-1}} - \sqrt{\frac{x}{qq-1}} = d$
10 x 11	14	$ae = \sqrt{\frac{qqxx}{qqq-2qq+1}} = p$
8 + 9	15	$aa + ee = \frac{qqx - x}{qq-1} = z$

Question 15. When z and x are given, to find the rest.

Viz. {	1	$aa + ee = z = 47232$	} Quære $a . e . \&c.$
	2	$aa - ee = x = 46080$	
1 + 2	3	$2aa = z + x$	
$3 \div 2$	4	$aa = \frac{z+x}{2}$	
1 - 2	5	$2ee = z - x$	
$5 \div 2$	6	$ee = \frac{z-x}{2}$	
$4 \omega^2$	7	$a = \sqrt{\frac{z+x}{2}}$	
$6 \omega^2$	8	$e = \sqrt{\frac{z-x}{2}}$	
7 + 8	9	$a + e = \sqrt{\frac{z+x}{2}} + \sqrt{\frac{z-x}{2}} = s$	
7 - 8	10	$a - e = \sqrt{\frac{z+x}{2}} - \sqrt{\frac{z-x}{2}} = d$	
7 x 8	11	$ae = \sqrt{\frac{zz - xx}{4}} = p$	
$7 \div 8$	12	$\frac{a}{e} = \frac{\sqrt{\frac{z+x}{2}}}{\sqrt{\frac{z-x}{2}}} = q$	

These fifteen Questions are proposed in Dr Pell's *Algebra*; but he pursues only the first Question throughout, and breaks off in the other fourteen, after the Values of what I call a and e are found. But I have proceeded in every one of them, to find the Values of all the unknown Qualities, because they afford

afford such Variety, as being well observed by a Learner, will be found very useful in the Solution of most Questions.

Note, I have chose to use the same Numbers for the respective Value of each Quantity throughout all the Questions, because they will be more satisfactory in proving the Work than various Numbers would have been. Not but that any Numbers may be taken at Pleasure, provided that the Number represented by a , be greater than that by e , &c. I have omitted the Numerical Calculations purely for the Learner to practise on.

Question 16. There are two Numbers, the Sum of their Squares is 2368; and the greater of them is in Proportion to the less, as 6 to 1. What are these Numbers?

Let a = the greater Number, e = the lesser, and $z = 2368$.

Then	1	$aa + ee = z$	} by the Question.
And	2	$a : e :: 6 : 1$	
	3	$1a = 6e$	
$2 \therefore$	4	$aa = 36ee$	
$3 \text{ } \textcircled{6}^2$	5	$ee = z - 36ee$	
$1 - 4$	6	$37ee = z$	
$5 + 36ee$	7	$ee = \frac{z}{37} = 64$	
$6 \div 37$	8	$e = \sqrt{\frac{z}{37}} = 8$	
$7 \text{ } \text{us}^2$	9	$6e = 6 \sqrt{\frac{z}{37}} = 48$	
8×6	10	$a = 48$	
$3 \text{ and } 9$			

Proof {

If $a =$	48
and $e =$	8
$aa =$	2304
$ee =$	64
$aa + ee =$	2368
and $48 : 8 :: 6 : 1$	

Question 17. There are three Numbers in continued Proportion, the Sum of the Extreams is 156, and the Mean is 72; What are the two Extreams?

That is, Suppose $a . m . e$ in $\div$, and $m = 72$.

Then {	1	$a + e = 156 = s$	} by the Question.
	2	$a : m :: m : e$	
	3	$ae = mm$	
$2 \therefore$	4	$aa + 2ae + ee = ss$	
$1 \text{ } \textcircled{6}^2$	5	$4ae + 4mm$	
3×4			

4 — 5	6	$aa - 2ae + ee = ss - 4mm$	
6 uv^2	7	$a - e = \sqrt{ss - 4mm}$	
1 + 7	8	$2a = s + \sqrt{ss - 4mm}$	
8 ÷ 2	9	$a = \frac{s + \sqrt{ss - 4mm}}{2} = 108$	} Or { $a = 48$ $e = 108$
1 — 9	10	$e = \frac{s - \sqrt{ss - 4mm}}{2} = 48$	

Question 18. There are three Numbers in $\div$, their Sum is 74, and the Sum of their Squares is 1924; What are those Numbers?

That is, a, e, y are in $\div$

Then {	1	$a + e + y = s = 74$	} Quære a, e, y .
	2	$aa + ee + yy = z = 1924$	
	3	$a : e :: e : y$	
5 $\therefore$	4	$ay = ee$	
1 — e	5	$a + y = s - e$	
2 — ee	6	$aa + yy = z - ee$	
4 × 2	7	$2ay = 2ee$	
6 + 7	8	$aa + 2ay + yy = z + ee$	
5 $\odot^2$	9	$aa + 2ay + yy = ss - 2se + ee$	
8 and 9	10	$z + ee = ss - 2se + ee$	
10 —	11	$2se = ss - z$	
11 ÷ 2	12	$e = \frac{ss - z}{2s} = 24$	
5,	13	$a + y = s - e = 50$	
13 $\odot^2$	14	$aa + 2ay + yy = 2500$	
4 × 4	15	$4ay = 4ee = 2304$	
14 — 15	16	$aa - 2ay + yy = 196$	
16 uv^2	17	$a - y = \sqrt{196} = 14$	
13 + 17	18	$2a = 50 + 14 = 64$	
18 ÷ 2	19	$a = 32$	} Or { $a = 18$ $y = 32$
13 — 19	20	$y = 50 - 32 = 18$	

Note, In all Questions about continual Proportionals, (either Arithmetical or Geometrical) where three Terms are sought, the Mean is the easiest found first (as above) and if all the Terms be Affirmative, then it is equal whether the first or last Term be the greatest.

Question

Question 19. There are three Numbers in $\div\div$ their Sum is 76; and if the Sum of the Extremes be multiplied into the Mean, that Product will be 1248; What are those Numbers?

<i>Viz.</i> {	1	$a : e :: e : y$	} by the Question.
	2	$a + e + y = s = 76$	
	3	$ae + ye = p = 1248$	
1 $\therefore$	4	$ay = ee$	
1 $\times e$	5	$ae + ee + ye = se$	
5 $- 3$	6	$ee = se - p$	
6 $- se$	7	$ee - se = -p$	
7 $C \square$	8	$ee - se + \frac{1}{4}ss = \frac{1}{4}ss - p$	
8 aw^2	9	$e - \frac{1}{2}s = \sqrt{\frac{1}{4}ss - p}$	
9 $+\frac{1}{2}s$	10	$e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = \left\{ \begin{array}{l} 52. \text{ per Theorem 3.} \\ 24 \text{ Chap. 8.} \end{array} \right.$	
2 $- 10$	11	$a + y = 52$	
4 $\times \frac{1}{4}$	12	$4ay = ee = 2304$	
11 $\odot^2$	13	$aa + 2ay + yy = 2704$	
13 $- 12$	14	$aa - 2ay + yy = 400$	
14 aw^2	15	$a - y = \sqrt{400} = 20$	
11 $+\frac{1}{2}$	16	$2a = 52 + 20 = 72$	
16 $\div 2$	17	$a = 36$	} } Or $a = 16$ and $y = 36$
11 $- 17$	18	$y = 52 - 36 = 16$	

N. B. If you take $e = \frac{1}{2}s + \sqrt{\frac{1}{4}ss - p} = 52$ (at the 10th Step) then it will be $76 - 52 = 24 = a + y$, which is impossible, *viz.* that the Mean should be greater than the Sum of the two Extreams. Therefore it must be $e = \frac{1}{2}s - \sqrt{\frac{1}{4}ss - p} = 24$. (See page 201.)

Question 20. There are three Numbers in Arithmetical Progression, the first being added to twice the second, and three times the third, their Sum will be 62; and the Sum of all their Squares is 275; What are those Numbers?

Suppose	1	a, e, y in Arithmetical Progression.	
And {	2	$a + 2e + 3y = 62$	} by the Question.
	3	$aa + ee + yy = 275$	
Then	4	$a + y = 2e$, per Sect. 1. Chap. 6.	
2 $- 4$	5	$2e + 2y = 62 - 2e$	
5 $\div 2$	6	$e + y = 31 - e$	
6 $- e$	7	$y = 31 - 2e$	
4 $- 7$	8	$a = 4e - 31$	

$8 \odot^2$	9	$aa = 16ee - 248e + 961$
$7 \odot^2$	10	$yy = 961 - 124e + 4ee$
$9 + 10$	11	$aa + yy = 20ee - 372e + 1922$
$3 - 11$	12	$ee = 372e - 20ee - 1647$
$12 + 20ee$	13	$21ee = 372e - 1647$
$13 - 372e$	14	$21ee - 372e = -1647$
$14 \div 21$	15	$ee - \frac{124}{7}e = -\frac{142}{7}$
$15 \odot \square$	16	$ee - \frac{124}{7}e + \frac{3844}{49} = \frac{3844}{49} - \frac{142}{7} = \frac{1}{49}$
$16 \omega^2$	17	$e - \frac{62}{7} = \sqrt{\frac{1}{49}} = \frac{1}{7}$
$17 + \frac{62}{7}$	18	$e = \frac{62}{7} \pm \frac{1}{7} = 9$, or $8 \frac{5}{7}$ the Mean
18×4	19	$4e = 36$, or $\frac{146}{7}$
8 and 19	20	$a = 36 - 31 = 5$, or $34 \frac{6}{7} - 31 = 3 \frac{6}{7}$
$18 \times \frac{1}{2}$	21	$2e = 18$, or $17 \frac{3}{7}$
7 and 21	22	$y = 31 - 18 = 13$, or $31 - 17 \frac{3}{7} = 13 \frac{4}{7}$

Question 21. There are three Numbers in Arithmetical Progression; the Square of the first Term being added to the Product of the other two is 576; the Square of the Mean being added to the Product of the two Extrems, make 612; and the Square of the last Term being added to the Product of the first into the second, is 792: What are those Numbers?

Suppose	1	a, e, y in Arith. Progreff. as before.
Then	2	$aa + ye = 576$
	3	$ee + ya = 612$
	4	$yy + ae = 792$
		} by the Question.
$1 \therefore$	5	$a + y = 2e$, per Sect. 1. Chap. 6.
$5 \times e$	6	$ae + ye = 2ee$
$2 + 4$	7	$aa + ye + yy + ae = 1368$
$7 - 6$	8	$aa + yy = 1368 - 2ee$
$3 - ee$	9	$ya = 612 - ee$
9×2	10	$2ya = 1224 - 2ee$
$8 + 10$	11	$aa + 2ya + yy = 2592 - 4ee$
$5 \odot^2$	12	$aa + 2ya + yy = 4ee$
11 and 12	13	$4ee = 2592 - 4ee$
$13 + 4ee$	14	$8ee = 2592$
$14 \div 8$	15	$ee = 324$
$15 \omega^2$	16	$e = \sqrt{324} = 18$, the Mean
8,	17	$aa + yy = 1368 - 2ee = 720$
10,	18	$2ya = 1224 - 2ee = 576$
17 - 18,	19	$aa - 2ya + yy = 720 - 576 = 144$

$$\begin{array}{l|l|l}
 1 \text{ } w^2 & 20 & a - y = \sqrt{144} = 12 \\
 5 + 20 & 21 & 2a = 2e + 12 = 48 \\
 21 \div 2 & 22 & a = 24 \\
 5 - 22 & 23 & y = 2e - 24 = 12
 \end{array}
 \quad \text{Or } \begin{cases} a = 12 \\ y = 24 \end{cases}$$

Question 22. It is required to find two such Numbers, that the Sum of their Squares may be $8226\frac{1}{2}$; and their Product being added to the Square of the lesser, may be $6921\frac{1}{2}$.

$$\begin{array}{l|l|l}
 \text{Viz. } \left\{ \begin{array}{l} 1 \\ 2 \end{array} \right. & \begin{array}{l} aa + ee = 8226\frac{1}{2} \\ ae + ee = 6921\frac{1}{2} \end{array} & \left. \begin{array}{l} \\ \end{array} \right\} \text{Quære } a \text{ and } e \\
 1 - 2 & 3 & aa - ae = 1305 \\
 3 + & 4 & ae = aa - 1305 \\
 4 \div a & 5 & e = \frac{aa - 1305}{a} \\
 5 \odot^2 & 6 & ee = \frac{a^4 - 2610aa + 1703025}{aa} \\
 1 - aa & 7 & ee = 8226,5 - aa \\
 6 \text{ and } 7 & 8 & \frac{a^4 - 2610aa + 1703025}{aa} = 8226,5 - aa \\
 8 \times aa & 9 & a^4 - 2610aa + 1703025 = 8226,5aa - a^4 \\
 9 + a^4 & 10 & 2a^4 - 2610aa + 1703025 = 8226,5aa \\
 10 + & 11 & 2a^4 - 10836,5aa = -1703025 \\
 11 \div 2 & 12 & a^4 - 5418,25aa = -85151,25 \\
 12 \odot \square & 13 & a^4 - 5418,25aa + 7339358,26562 = 6487845,765 \\
 13 w^2 & 14 & aa - 2709,125 = \sqrt{6487845,765625 - 2547,125} \\
 14 + 27\&c. & 15 & aa = 2709,125 + 2547,125 \\
 Suppose & 16 & aa = 2709,125 + 2547,125 = 5256,25 \\
 Then & 17 & a = \sqrt{5256,25} = 72,5 \\
 And 5, & 18 & e = \frac{aa + 1305}{a} = \frac{5256,25 - 1305}{72,5} = 54,5 \\
 Or let & 19 & aa = 2709,125 - 2547,125 = 162 \\
 10 w^2 & 20 & a = \sqrt{162} = 12,72 \&c. \\
 Then & 21 & e = \frac{162 - 1305}{12,72}, \text{ which is impossible.} \\
 Therefore & & a = 72,5 \} \text{ as at the 17th and 18th Steps.} \\
 And & & e = 54,5 \}
 \end{array}$$

This Question may be performed with less Trouble, by substituting Letters from the known Numbers.

$$\text{Viz. } \begin{cases} aa + ee = z \\ ae + ee = p \end{cases} \text{ Then let } z - p = d = aa - ae, \&c.$$

F f

Question

Question 23. It is required to find three such Numbers, that the Sum of the first and second, being multiplied with the third, may be 37824; and the Sum of the second and third, multiplied with the first, may be 59944; also, that the Sum of the first and third, being multiplied with the second, may be 52456.

Let a, e, y represent the three Numbers.

Then	{	1	$ay + ey = 37824 = b$	} Quære a, e, y .
		2	$ea + ya = 59944 = c$	
		3	$ae + ye = 52456 = d$	
1 + 2 + 3	4	$2ae + 2ay + 2ye = b + c + d$		
Let	5	$z = b + c + d$		
4 ÷ 2	6	$ae + ay + ye = \frac{1}{2}z = \frac{b + c + d}{2}$		
6 - 3	7	$ay = \frac{1}{2}z - d = \frac{z - 2d}{2}$		
7 ÷ a	8	$y = \frac{z - 2d}{2a}$		
6 - 2	9	$ye = \frac{1}{2}z - c = \frac{z - 2c}{2}$		
6 - 1	10	$ae = \frac{1}{2}z - b = \frac{z - 2b}{2}$		
10 ÷ a	11	$e = \frac{z - 2b}{2a}$		
8 × 11	12	$ye = \frac{z - 2d}{2a} \times \frac{z - 2b}{2a} = \frac{zz - 2dz - 2bz + 4bd}{4aa}$		
9 and 12	13	$\frac{z - 2c}{2} = \frac{zz - 2dz - 2bz + 4bd}{4aa}$		
13 × 4aa	14	$2zaa - 4caa = zz - 2dz - 2bz + 4bd$		
14 ÷	15	$aa = \frac{zz - 2dz - 2bz + 4bd}{2z - 4c} = 55696$		
15 $\sqrt{}$	16	$a = \sqrt{55696} = 236$		
11	17	$e = \frac{z - 2b}{2a} = 158$		
8	18	$y = \frac{z - 2d}{2a} = 96$		

Question 24. It is required to find two such Numbers, that their Sum being subtracted from the Sum of their Squares, may leave 14, and if their Product be added to their Sum, it may make 14.

Let a and e be put for the Numbers, and let $y = a + e$

Then $\left\{ \begin{array}{l} 1 \mid aa + ee - y = 14 \\ 2 \mid ae + y = 14 \end{array} \right\}$ by the Question.

1 + 7	3	$aa + ee = 14 + y$
2 - y	4	$ae = 14 - y$
$4 \times \frac{1}{2}$	5	$2ae = 28 - 2y$
3 + 5	6	$aa + 2ae + ee = 42 - y$
$6w^2$	7	$a + e = \sqrt{42 - y}$
But	8	$a + e = y$, by Substitution above.
7 and 8	9	$y = \sqrt{42 - y}$
$9 \odot^2$	10	$yy = 42 - y$
$10 + y$	11	$yy + y = 42$
11 C □	12	$yy + y + \frac{1}{4} = 42 + \frac{1}{4} = 42,25$
$12w^2$	13	$y + \frac{1}{2} = \sqrt{42,25} = 6,5$
$13 - \frac{1}{2}$	14	$y = 6,5 - \frac{1}{2} = 6$
Consequent	15	$a + e = 6$, by Restitution from above.
3 and 14	16	$aa + ee = 14 + 6 = 20$
5 and 15	17	$2ae = 28 - 12 = 16$
16 - 17	18	$aa - 2ae + ee = 4$
$18w^2$	19	$a - e = \sqrt{4} = 2$
15 + 19	20	$2a = 8$
$23 \div 2$	21	$a = 4$
15 - 21	22	$e = 6 - 4 = 2$

Proof {

If $a = 4$, and $e = 2$

Then $aa + ee - a - e = 14$

And $ae + a + e = 14$

According to the Question.

Question 25. Three Men discoursing of their Money; saith the first, if 100 *l.* were added to my Money, it would be as much as both your Money put together; said the second Man, if 100 *l.* were added to my Money, I should have twice as much as both you have; saith the third Man, if 100 *l.* were added to my Money, I should have then three times as much Money as both you have: How much Money had each Man?

Let *a* represent the first Man's Money, *e* the second, and *y* the third.

Then {	1	$a + 100 = e + y$	} by the Question.
	2	$e + 100 = 2a + 2y$	
	3	$y + 100 = 3a + 3e$	
1 - a	4	$e + y - a = 100 = s$	} Quære <i>a</i> , <i>e</i> , <i>y</i> .
2 - e	5	$2a + 2y - e = 100 = s$	
3 - y	6	$3a + 3e - y = 100 = s$	
4 and 6	7	$e + y - a = 3a + 3e - y$	
$7 +$	8	$2y = 4a + 2e$	
$5 - 8$	9	$2a - e = s - 4a - 2e$	
$9 + 4a - 2e$	10	$6a + e = s = 100$	
$4 + 6$	11	$2a + 4e = 2s = 200$	

10×4	12	$24a + 4e = 4s = 400$
$12 - 11$	13	$22a = 2s = 200$
$13 \div 22$	14	$a = \frac{s}{11} = \frac{100}{11} = 9 \frac{1}{11} l.$
$10 - 6a$	15	$e = s - 6a = 100 - \frac{600}{11} = \frac{500}{11} = 45 \frac{5}{11} l.$
$8 \div 2$	16	$y = 2a + e = \frac{200}{11} + \frac{500}{11} = \frac{700}{11} = 63 \frac{7}{11} l.$

Answer. The $\left\{ \begin{array}{l} \text{first} \\ \text{second} \\ \text{third} \end{array} \right\}$ Man had $\left\{ \begin{array}{l} 9 l. \quad 1 s. \quad 9 \frac{2}{11} d. \\ 45 l. \quad 9 s. \quad 1 \frac{2}{11} d. \\ 64 l. \quad 12 s. \quad 8 \frac{1}{11} d. \end{array} \right.$

Question 26. Three Men have each such a Sum of Money, that if the first and second Mens Money be added to Half of what the third Man hath; that Sum will be 92 *l.* And if the second and third Mens Money be added to one third Part of the first Man's Money, that Sum will be 92 *l.* Lastly, if one fourth Part of the second Man's Money be added to the first and third Man's Money, that Sum will also be 92 *l.* How much was each Man's Money?

Put *a* for the 1st Man's Money, *e* for the 2d, and *y* for the 3d.

Then $\left\{ \begin{array}{l} 1 \\ 2 \\ 3 \end{array} \right. \left. \begin{array}{l} a + e + \frac{1}{2}y = s \\ \frac{1}{3}a + e + y = s \\ \frac{1}{4}e + a + y = s \end{array} \right\}$ by the Question; and $s = 92$

1 and 2	4	$a + e + \frac{1}{2}y = \frac{1}{3}a + e + y$
$4 - e$	5	$a + \frac{1}{2}y = \frac{1}{3}a + y$
$5 \times \frac{2}{3} \times 3$	6	$6a + 3y = 2a + 6y$
$6 \div 4$	7	$4a = 3y$
2×3	8	$a + 3e + 3y = 3s$
$8 - 7$	9	$a + 3e = 3s - 4a$
$9 - a$	10	$3e = 3s - 5a$
$10 \div 3$	11	$e = \frac{3s - 5a}{3}$
3×4	12	$e + 4a + 4y = 4s = 368$
$12 - 2$	13	$3\frac{2}{3}a + 3y = 3s = 276$
13 and 7	14	$3\frac{2}{3}a + 4a = 3s = 276$
14×3	15	$11a + 12a = 9s = 828$
$15 \div 23$	16	$a = \frac{9s}{23} = \frac{828}{23} = 36 l. \text{ the 1st Man's Money.}$
	17	$e = \frac{3s - 5a}{3} = \frac{276 - 180}{3} = 32 l. \text{ the 2d Man's Money.}$
$7 \div 3$	18	$y = \frac{4a}{3} = \frac{144}{3} = 48 l. \text{ the 3d Man's Money.}$

Question

Question 27. Four Men walking abroad, found a Purse of Shillings only, out of which every one took a Number at an Adventure; afterwards by comparing their Numbers together they found, that if the first took 25 Shillings from the second, it would make his Number equal with what the second had then left; if the second took 30 Shillings from the third, his Money would then be triple to what the third had left, and if the third took 40 Shillings from the fourth, his Money would then be double to what the fourth had left; lastly, the fourth taking 50 Shillings from the first, he would then have three times as much as the first had left, and 5 Shillings more: It is required to tell how many Shillings each Man had.

Put a for the first Sum, e the second, y the third, and u the fourth.

Then	{	1	$a + 25 = e - 25$	} by the Question.
		2	$e + 30 = 3y - 90$	
		3	$y + 40 = 2u - 80$	
		4	$u + 50 = 3a - 145$	
$1 + 25$	5	$a + 50 = e$		
$2 - 30$	6	$3y - 120 = e$		
$5 \text{ and } 6$	7	$a + 50 = 3y - 120$		
$7 + 120$	8	$a + 170 = 3y$		
$8 \div 3$	9	$y = \frac{a + 170}{3}$		
$3 - 40$	10	$y = 2u - 120$		
$9 \text{ and } 10$	11	$2u - 120 = \frac{a + 170}{3}$		
$1 + 120$	12	$2u = \frac{a + 170}{3} + 120 = \frac{a + 530}{3}$		
$12 \div 2$	13	$u = \frac{a + 530}{6}$		
$4 - 50$	14	$u = 3a - 195$		
$13 \text{ and } 14$	15	$3a - 195 = \frac{a + 530}{6}$		
15×6	16	$18a - 1170 = a + 530$		
$16 \div$	17	$17a = 1700$		
$17 \div 17$	18	$a = 100 \text{ the } 1^{st}$		
by the 5	19	$e = 150$	$2d$	} Man's Number of Shillings.
by the 9	20	$y = 90$	$3d$	
by the 14	21	$u = 105$	$4th$	

Question

Question 28. Four Men have each a Sum of Money, which being put all together makes 250 Pounds; and if to the first Man's Money be added 8 Pounds, it will be just as much as the second Man's Money decreased by 8 Pounds, and as much as 8 times the third Man's Money, and but as much as one eighth Part of the fourth Man's Money; how much had each Man?

Let a, e, y, u , represent the four Men's Money.

Then $\left\{ \begin{array}{l} 1 \quad a + e + y + u = s \\ 2 \quad a + b = e - b \\ 3 \quad yb = \frac{u}{b} = a + b \end{array} \right\}$ by the Question. Let $s = 250$ and $b = 8$, or any other Number at Pleasure.

$$\begin{array}{ll} 2 + b & 4 \quad a + 2b = e \\ 3 \div b & 5 \quad y = \frac{a + b}{b}, \text{ because } yb = a + b \\ 3 \times b & 6 \quad u = ba + bb, \text{ for } \frac{u}{b} = a + b \\ 4 + 5 + 6 & 7 \quad e + y + u = a + 2b + \frac{a + b}{b} + ba + bb \\ 1 - a & 8 \quad e + y + u = s - a \\ 7 \text{ and } 8 & 9 \quad a + 2b + \frac{a + b}{b} + ba + bb = s - a \\ 9 \times b & 10 \quad ba + 2bb + a + b + bba + bbb = bs - ba \\ 10 + & 11 \quad 2ba + bba + a = bs - bbb - 2bb - b \\ 11 \div & 12 \quad a = \frac{bs - bbb - 2bb - b}{bb + 2b + 1} = 16,691358 \text{ \&c.} \\ \text{by the 4,} & 13 \quad e = a + 2b = 32,691358 \text{ \&c.} \\ \text{by the 5,} & 14 \quad y = \frac{a + b}{b} = 3,086419 \text{ \&c.} \\ \text{by the 6,} & 15 \quad u = ba + bb = 197,530864 \text{ \&c.} \end{array}$$

That is, $\left\{ \begin{array}{lll} l. & s. & d. \\ a = & 16 . 13 . & 9,92592 \\ e = & 32 . 13 . & 9,92592 \\ y = & 3 . 1 . & 8,74056 \\ u = & 197 . 10 . & 7,40736 \end{array} \right.$

Consequently $a + e + y + u = 249 . 19 . 11,99976$ which should be just 250 $l.$ the Sum proposed in the Question. Now what it wants of that Sum, proceeds from the Imperfection of the Decimal Parts being not continued on to more Places, which would have brought it nearer the Truth, tho' not perhaps exactly so. *Sect. 5. Chap. 5. Part I.*

Question

Question 29. Several Merchants enter into Partnership, every one put into the Stock 65 times as many Pounds as there were Partners; with that Stock they traded and gained as many Pounds per 100*l.* as there were Partners. Now if 10*l.* 10*s.* be added to, and substracted from, their Gain, the Product of that Sum and Difference will be 6491*l.* 6*s.* 3*d.*

Quære, How many Merchants there were, &c.

Let	1	$a =$ the Number of Merchants.
1×65	2	$65a =$ every one's Sum put into Stock.
$2 \times a$	3	$65aa =$ the whole Stock.
And	4	$100 : a :: 65aa : \frac{65aaa}{100}$, by the Question.
<i>Viz.</i>	5	$\frac{65aaa}{100} =$ the whole Gain.
$5 + 10,5$	6	$\frac{65aaa}{100} + 10,5$
$5 - 10,5$	7	$\frac{65aaa}{100} - 10,5$
6×7	8	$\frac{4225aaaaaa}{10000} - 110,25 = 6491,3125$, by the Quest.
8×10000	9	$4225a^6 - 1102500 = 64913125$
$9 +$	10	$4225a^6 = 66015625$
$10 \div 4225$	11	$a^6 = \frac{66015625}{4225} = 15625$
$11 \sqrt[6]{}$	12	$a = \sqrt[6]{15625} = 5$ the Number of Merchants.
12×65	13	$65a = 325$ the Number of Pounds each put in.

Question 30. Three Merchants join Stocks together; the first Man's Stock was less than the second Man's by 13*l.* the second and third Man's Stock was 175*l.* in trading they gain 48*l.* more than their whole Stock was; the first Man's proportional Part of the Gain was 78. What was each Man's Stock and Part of the Gain?

Let a, e, y represent each Man's Stock.

Then	{	1		$a + e + y = s$	the whole Stock.
		2		$s + 48 =$	the whole Gain.
And	{	3		$a + 13 = e$	} by the Question.
		4		$e + y = 175$	
$4 + a$		5		$a + e + y = 175 + a$	
1 and 5	6	$s = 175 + a$			

6 and 2

6 and 2	7	$s + 48 = 223 + a$
But	8	$175 + a : 223 + a :: a : 78$ per Question.
8 $\therefore$	9	$aa + 223a = 78a + 13650$
9 — 78a	10	$aa + 145a = 13650$
10 C $\square$	11	$aa + 145a + 5256,25 = 18906,25$
11 uw^2	12	$a + 72,5 = \sqrt{18906,25} = 137,5$
12 — 72,5	13	$a = 137,5 - 72,5 = 65$
3,	14	$e = a + 13 = 78$
4 — 14	15	$y = 97$
Then	16	$65 : 78 :: 78 : 93l. 12s. = e's \text{ Gain.}$
Again	17	$65 : 78 :: 97 : 116l. 8s. = y's \text{ Gain.}$
Proof {	18	$116l. 8s. + 93l. 12s. + 78l. = 288l. \text{ the Gain.}$
	19	$65 + 78 + 97 = 240. \text{ the whole Stock.}$
18 — 19	20	$288 - 240 = 48 \text{ the Gain more than the Stock.}$

Question 31. A Father at his Death left his three Sons his Money in this manner; to the eldest he gave half of it, wanting 44 Pounds; to the second he gave one third of it, and 14 Pounds more; to the youngest he gave the Remainder, which was less than the Share of the second Son, by 82 Pounds: What was each Son's Share?

Let a, e, y be the three Shares, and $z =$ the whole Sum.

Then {	1	$a + e + y = z$	} by the Question.
	2	$a = \frac{1}{2}z - 44$	
	3	$e = \frac{1}{3}z + 14$	
	4	$y = \frac{1}{3}z + 14 - 82$	
2 + 3 + 4	5	$a + e + y = \frac{2z}{3} + \frac{z}{2} - 98$	
1 and 5	6	$z = \frac{2z}{3} + \frac{z}{2} - 98$	
6 $\times$ 3	7	$3z = 2z + \frac{3z}{2} - 294$	
7 $\times$ 2	8	$6z = 4z + 3z - 588$	
8 +	9	$z = 588, \text{ the whole Sum that was left.}$	
2 and 9	10	$a = \frac{1}{2}z - 44 = 250, \text{ the eldest Son's Share.}$	
3 and 9	11	$e = \frac{1}{3}z + 14 = 210, \text{ the second Son's Share.}$	
4 and 9	12	$y = \frac{1}{3}z + 14 - 82 = 128, \text{ the youngest \&c.}$	

Question 32. A Man playing at Hazard or Dice, won the first Throw just so much Money as he had in his Pocket; the second

second Throw he won the Square Root of what he then had, and five Shillings more; the third Throw he won the Square of all he then had; after which his whole Sum was 112*l.* 16*s.* What Money had he when he began to play?

Suppose	1	$a =$ his first Sum. Then
1×2	2	$2a =$ his Sum after the first Throw.
And	3	$5 + \sqrt{2a} =$ the Winnings at the 2d Throw.
$2 + 3$	4	$2a + 5 + \sqrt{2a} =$ the Sum after the 2d Throw.
$4 \odot^2$	5	$4aa + 22a + 25 + 4a\sqrt{2a} + 10\sqrt{2a} =$ the Winnings at the 3d Throw; and therefore
$4 + 5$	6	$4aa + 24a + 30 + 4a\sqrt{2a} + 11\sqrt{2a} = 2256$ Shil.

But to avoid these Surd Quantities, let us, instead of supposing $a =$ the first Sum, make a second Trial, viz.

Let	1	$2aa =$ the first Sum.
1×2	2	$4aa =$ the Sum after the first Throw.
Then	3	$2a + 5 =$ the Sum won at the 2d Throw.
$2 + 3$	4	$4aa + 2a + 5 =$ his Sum after the 2d Throw.
$4 \odot^2$	5	$16a^4 + 16a^3 + 44aa + 20a + 25 =$ the Winnings at the 3d Throw; and therefore
$4 + 5$	6	$16a^4 + 16a^3 + 48aa + 22a + 30 = 2256$ Shil.

Yet again, to avoid these high Equations, let us make a third Supposition; thus,

Let	1	$\frac{aa}{2} =$ the first Sum.
1×2	2	$aa =$ the Sum after the first Throw.
Then	3	$a + 5 =$ the Winnings at the 2d Throw.
$2 + 3$	4	$aa + a + 5 =$ the Sum after the 2d Throw.
Substi.	5	$e = aa + a + 5.$
$5 \odot^2$	6	$ee =$ the Winnings at the 3d Throw. Then
$5 + 6$	7	$ee + e = 2256$ Shillings by the Question.
$7 C \square$	8	$ee + e + 0,25 = 2256,25$
$8 uv^2$	9	$e + 0,5 = \sqrt{2256,25} = 47,5$
$9 - 0,5$	10	$e = 47$
5 and 10	11	$aa + a + 5 = 47$
$11 - 5$	12	$aa + a = 42$
12, $C \square$	13	$aa + a + 0,25 = 42,25$
$1 uv^2$	14	$a + 0,5 = \sqrt{42,25} = 6,5$
$14 - 0,5$	15	$a = 6$
$15 \odot^2$	16	$aa = 36$
$16 \div 2$	17	$\frac{aa}{2} = \frac{36}{2} = 18$ { The Shillings he had in his Pocket when he began to play.

G g

Note,

Note, In resolving of the last Question, I have made three different Suppositions for the Thing sought, purely as an Instance, to shew the young Learner how well he ought to consider the Nature of the Question, when he first states it, and make choice of representing the Thing sought, so as to avoid running it into Surds, if possible, viz. as in the first Supposition of $a =$ the first Sum, &c. Not but that such Equations may be solved, as shall be shewed in the next Chapter. However, it is most like an Artist to perform Things of this Nature the nearest and easiest way they can be done.

Question 33. Suppose there were two equal Circles, whose Peripheries (viz. Circumferences) are divided into 44310 equal Parts; and that those Circles were so placed upon one Axis, as to move the contrary way to each other; and suppose one of them to move but one of these equal Parts the first Day, two Parts the second Day, three Parts the third Day, and so on in Arithmetical Progression, viz. 1, 2, 3, 4, 5, &c. and the other to move every Day the Cube of those Parts, 1, 8, 27, 64, 125, &c. of the same Parts: How many Parts and how many Days must each Circle move, before the same two Points meet that were together when they began to move?

In order to give a ready Solution to this Question (or any other in this Kind) it will be convenient to premise this Lemma.

L E M M A.

The Sum of any Series of Cubes whose Roots are in Arithmetick Progression (the first Term, and common Difference being Unity or 1) is equal to the Square of the Sum of all those Roots.

As in these

Terms in Arith. their Cubes.
 &c.

1	1
2	8
3	27
4	64
5	125
6	216 &c.

$$\underline{21} \times \underline{21} = \underline{441} \text{ Sum of their Cubes.}$$

Let $\left| \begin{array}{l} 1 \\ 2 \\ 3 \\ 4 \end{array} \right|$ $a =$ the Sum of all the Parts the 1st Circle moves.
 Then $\left| \begin{array}{l} 2 \\ 3 \\ 4 \end{array} \right|$ $aa =$ the Sum of all the Parts the 2d moves.
 Consequen. $\left| \begin{array}{l} 3 \\ 4 \end{array} \right|$ $aa + a = 44310$ by the Quest. (per Lem.
 $2C \square \left| \begin{array}{l} 4 \end{array} \right|$ $aa + a + 0,25 = 44310,25$

4 us²

$$\begin{array}{r|l}
 4 \omega^2 & 5 \mid a + 0,5 = \sqrt{44310,25} = 210,5 \\
 5 - 0,5 & 6 \mid a = 210 \left\{ \begin{array}{l} \text{the Number of Parts the first Circle} \\ \text{must move.} \end{array} \right. \\
 6 \odot^2 & 7 \mid aa = 44100 \left\{ \begin{array}{l} \text{the Number of Parts the second} \\ \text{Circle moves.} \end{array} \right.
 \end{array}$$

Next to find the Number of Days they moved; there is given the first Term = 1, the common Difference = 1, and the Sum of all the Terms = 210, thence to find the last Term, which in this Case is the same with the Number of all the Terms.

Let $a=1$ the first Term, $e=1$ the common Difference, and $s=210$ the Sum of all the Terms, to find y = the last Term; as per Sect. 1. Chap. 6. Then $yy + ey = 2s + aa - ae$ by the 16 Step, Page 186; that is, $yy + y = 210 \times 2 = 420$ &c. Hence $y = 20$ the Number of Days required.

I shall now proceed to give an Example or two of the Method used in arguing about unlimited Questions; viz. such Questions which admit of various Answers, such as those in *Alligation Alternate* promised in Page 117.

In order to shorten that Work, it will be convenient for the Learner to know the two Signs of Comparison, $>$ and $<$. The Sign $>$ is of **Greater than**; as $b > a$ signifies that b is greater than a . The Sign $<$ is of **Lesser than**; as $b < d$ signifies that b is lesser than d , &c.

EXAMPLE I.

Question 34. A Tobacconist hath three Sorts of Tobacco, viz. one of 2s. 8d. the Pound, another of 20d. the Pound, and a third Sort of 16d. the Pound; of these he would make a Mixture to contain 56 Pound, that may be sold for 22d. the Pound: How much of each Sort may he take?

Let a = the Quantity of that worth 32 Pence the Pound, e = that of 20 Pence the Pound, and y = that of 16 Pence the Pound;

$$\begin{array}{l}
 \text{Then } a + e + y = 56 \\
 \text{And } 32a + 20e + 16y = 1232
 \end{array}
 \left. \vphantom{\begin{array}{l} a + e + y = 56 \\ 32a + 20e + 16y = 1232 \end{array}} \right\} \begin{array}{l} \text{viz. each Quantity multipli-} \\ \text{ed into it's own Price, equals} \\ \text{their Sum multiplied into the} \\ \text{mean Price.} \end{array}$$

G g 2

This

This Question being thus stated, it appears by Rule 1, Page 176, that it is capable of innumerable Answers; because for any one of these three Letters, a, e, y , there may be taken any Number at Pleasure, provided it be less than 56. But although that may be truly done, yet there are several Ways of arguing about these Sorts of Questions, which will limit or bound them to all their proper or possible Answers in whole Numbers. Thus,

Let	1	$a + e + y = 56$	} as above.
And	2	$32a + 20e + 16y = 1232$	
1 — a	3	$e + y = 56 - a$	
2 — $32a$	4	$20e + 16y = 1232 - 32a$	
3 × 16	5	$16e + 16y = 896 - 16a$	
4 — 5	6	$4e = 336 - 16a$	
6 ÷ 4	7	$e = 84 - 4a$; hence $a < 21$	
3 — 7	8	$5 = 3a - 28$; hence $a > \frac{28}{3}$ or $9\frac{1}{3}$	

From the two last Steps it appears, that the Quantity signified by a , ought to be less than 21, and greater than $9\frac{1}{3}$; that is, any Number betwixt $9\frac{1}{3}$ and 21, may be taken for the Value of a : Consequently there may be eleven Answers to this Question in whole Numbers.

Suppose $a = 10$, then $e = 84 - 40 = 44$, per 7th Step; and $y = 30 - 28 = 2$, per 8th Step. Again, if $a = 11$, then $e = 84 - 44 = 40$, per 7th Step; and $y = 33 - 28 = 5$, per 8th Step: and so on for the rest, which will be as in the following Table.

a	e	y	a	e	y	a	e	y
10	44	2	14	28	14	18	12	26
11	40	5	15	24	17	19	8	29
12	36	8	16	20	20	20	4	32
13	32	11	17	16	23			

Thus it will be easy to find out and collect all the limited Answers to any Question (of this Kind) wherein there are only three Quantities proposed to be mixed: But when there are more than three, then the Work requires a little more Trouble; because the single Limits of all the Quantities above two must be found; that is, if there are four Quantities concerned in the Question, the Limits of two of them must be found; if five Quantities are concerned, then the Limits of three of them must be found, &c. As in the following Question.

Question

Question 35. Suppose it were required to mix four Sorts of Wine together; viz. one Sort worth 7 s. 4 d. the Gallon, another Sort worth 4 s. 7 d. the Gallon, a third Sort worth 3 s. 8 d. the Gallon, and a fourth Sort worth 2 s. 9 d. the Gallon: How much of each Sort may be taken to make a Mixture of 63 Gallons, so as that the whole Quantity may be sold for 5 s. 6 d. the Gallon, without Loss, &c.

First, let all these several Rates, and the mean Rate, be reduced to one Denomination, viz. into Pence.

$$\text{Viz. } \left\{ \begin{array}{l} 7 \text{ s. } 4 \text{ d.} = 88 \text{ d.} \\ 4 \text{ s. } 7 \text{ d.} = 55 \text{ d.} \\ 3 \text{ s. } 8 \text{ d.} = 44 \text{ d.} \\ 2 \text{ s. } 9 \text{ d.} = 33 \text{ d.} \end{array} \right\} \text{ and } 5 \text{ s. } 6 \text{ d.} = 66.$$

Put a = the Quantity of that worth 88 d. the Gallon; e = that of 55 d. the Gallon, y = that of 44 d. the Gallon, and u = that of 33 d. the Gallon.

Then	1	$a + e + y + u = 63$ by the Question.
And	2	$88a + 55e + 44y + 33u = 4158 = 63 \times 66$
$1 - a$	3	$e + y + u = 63 - a$
$2 - 88a$	4	$55e + 44y + 33u = 4158 - 88a$
3×33	5	$33e + 33y + 33u = 2079 - 33a$
$4 - 5$	6	$22e + 11y = 2079 - 55a$
$6 \div 11$	7	$2e + y = 189 - 5a$; hence $a < \frac{189}{5}$ or $37\frac{4}{5}$
3×55	8	$55e + 55y + 55u = 3465 - 55a$
$8 - 4$	9	$11y + 22u = 33a - 693$
$9 \div 11$	10	$y + 2u = 3a - 63$; hence $a > \frac{63}{3}$ or 21

From the 7th and 10th Steps it appears, that the Quantity of that Sort of Wine denoted by a , must be less than $37\frac{4}{5}$ Gallons, and greater than 21 Gallons: that is, it may be a = any Number of Gallons betwixt 21 and $37\frac{4}{5}$. Whence it follows, that there may be collected 16 Answers to this Question from the Limits of a only.

Next to find the Limits of e , y , and u .

Suppose	11	$a = 22$, then will $5a = 110$, and $3a = 66$
But	12	$2e + y = 189 - 5a = 79$, per 7th Step.
$12 - 2e$	13	$y = 79 - 2e$; hence $e < \frac{79}{2}$ or $39\frac{1}{2}$
Again	14	$e + y + u = 63 - a = 41$, per 3d Step.
$14 - e$	15	$y + u = 41 - e$
$15 - 13$	16	$u = e - 38$; hence $e > 38$

From the 13th and 16th Steps it appears, that if $a = 22$, then $e = 39$, $y = 79 - 2e = 1$, and $u = e - 38 = 1$.

Again,

Again,

Suppose	17	$a = 23$, then $5a = 115$, and $3a = 69$
But	18	$2e + y = 189 - 5a = 74$, per 7th Step.
$18 - 2e$	19	$y = 74 - 2e$; hence $e < \frac{74}{2} = 37$
Again	20	$e + y + u = 63 - a = 40$, per 3d Step.
$20 - e$	21	$y + u = 40 - e$
$21 - 19$	22	$u = e - 34$, hence $e > 34$.

From the 19th and 22d Steps it appears, that if $a = 13$, then e may be either 35 or 36.

Once more for a further Illustration.

Let	23	$a = 24$, then $5a = 120$, and $3a = 72$
But	24	$2e + y = 189 - 5a = 69$, per 7th Step.
$24 - 2e$	25	$y = 69 - 2e$; hence $e < \frac{69}{2}$ or $34\frac{1}{2}$
Again	26	$e + y + u = 63 - a = 39$, per 3d Step.
$26 - e$	27	$y + u = 39 - e$
$27 - 25$	28	$u = e - 30$, hence $e > 30$.

From hence it appears, that if $a = 24$, then e may be either 31, 32, 33, or 34, viz. it may be any Number betwixt 30 and $34\frac{1}{2}$ by the 25th and 28th Steps; from whence the Values of y and u may be easily found.

$$\text{That is, if } \begin{cases} e = 31. & \text{then } y = 7. & \text{And } u = 1 \\ e = 32. & y = 5. & u = 2 \\ e = 33. & y = 3. & u = 3 \\ e = 34. & y = 1. & u = 4 \end{cases}$$

Proceeding on in this manner with all the other single Values of a , there may be found above 120 Answers to this Question in whole Numbers: and if you please to put $a =$ Fractions, there may be found an innumerable Set of Answers; whereas the Rule of *Alligation* in *Vulgar Arithmetick* affords but only one Answer in Fractions, to wit, that of $a = 31\frac{1}{2}$, $e = 10\frac{1}{2}$, $y = 10\frac{1}{2}$, $u = 10\frac{1}{2}$; as may be easily tried per Rule *Page 115, &c.*

These two Examples being well understood (especially if the last be thoroughly pursued) may suffice to shew the Method of limiting the Answers to all Sorts of Questions of this Kind. I shall therefore conclude this Chapter of Questions with giving a Solution to the Enigma (or Riddle) proposed (but not answered) by Mr *John Kersey*, in the Close of the *Appendix* to his *Arithmetick*, which

which affords several pretty Questions, the Solution whereof will discover a certain Sentence consisting of three Words, which must be found by the Help of Figures placed (or supposed to be placed) over the twenty-four Letters of the Alphabet.

Thus $\left\{ \begin{array}{l} 1 \cdot 2 \cdot 3 \cdot 4 \cdot 5 \cdot 6 \cdot 7 \cdot \&c. \text{ called Indices.} \\ a \cdot b \cdot c \cdot d \cdot e \cdot f \cdot g \cdot \&c. \text{ to the last Letter.} \end{array} \right.$

So that if the Index of that Letter be once found, the Letter to which it belongs is consequently known.

The Enigma.

1. If the Difference between the Indices of the second Letter of the second Word, and the third Letter of the first Word, be multiplied into the Difference of their Squares, the Product will be 576; and if their Sum be multiplied into the Sum of their Squares, that Product will be 2336, the Index of the said third Letter being the greatest.

Let	1	$a =$ the greater Index, or that of the 3d Letter.
And	2	$e =$ the lesser, or that of the 2d Letter.
Then	$\left\{ \begin{array}{l} 3 \\ 4 \end{array} \right.$	$\left. \begin{array}{l} a - e \times aa - ee = 576 \\ a + e \times aa + ee = 2336 \end{array} \right\}$ by the Question.
$3 \times$	5	$aaa - aae - aee + eee = 576$
$4 \times$	6	$aaa + aae + aee + eee = 2336$
$6 - 5$	7	$2aee + 2ee = 1760$
$6 + 7$	8	$aaa + 3aee + 3aee + eee = 4096$
$8 \sqrt[3]{}$	9	$a + e = \sqrt[3]{4096} = 16$
$4 \div a + e$	10	$aa + ee = \frac{2336}{a + e} = \frac{2336}{16} = 146$
$9 \odot^2$	11	$aa + 2ae + ee = 256$
$11 - 10$	12	$2ae = 110$
$10 - 12$	13	$aa - 2ae + ee = 36$
$13 \sqrt[3]{}$	14	$a - e = \sqrt[3]{36} = 6$
$9 + 14$	15	$2a = 22$
$15 \div 2$	16	$a = 11$
$9 - 16$	17	$e = 5$

$\left. \begin{array}{l} a = 11 \\ e = 5 \end{array} \right\} \left\{ \begin{array}{l} \text{From hence it appears, that the 3d} \\ \text{Letter of the 1st Word is l, and the} \\ \text{2d Letter of the 2d Word is e.} \end{array} \right.$

Note, In order to set down the Letters (as they become found) in their proper Places, it may be convenient to supply the vacant Places with Stars.

Thus $\left\{ \begin{array}{l} \text{First Word.} \\ **l** \end{array} \right. \quad \begin{array}{l} \text{Second Word.} \\ **e*** \end{array} \quad \begin{array}{l} \text{Third Word.} \\ ***** \end{array}$

2. The

2. The Indices last found, are the two Extrems of four Numbers in Arithmetical Progression, the lesser Mean being the Index of the first Letter of the third Word; and the greater Mean is the Index of the fourth and last Letter of the first Word. *Viz.* 5 . 7 . 9 . 11 are the four Terms in Arithmetical Progression. Whence it appears, that *G* (whose Index is 7) is the first Letter of the third Word; and that *i* (whose Index is 9) is the fourth or last Letter of the first Word; which being placed down, will stand thus,

$$* * l i * e * * * * . G * * * * .$$

3. The second Letter of the third Word is the same with the third Letter of the first Word; and the fifth Letter of the third Word is the same with the last Letter of the first Word: Whence the Letters will stand thus,

$$* * l i . * e * * * * . G l * * * * i * .$$

4. The Sum of the Squares of the Indices of the first and second Letters of the first Word is 520, and the Product of the same Indices is seven Ninths of the Square of the greater Index, which is the Index of the said first Letter.

Let a = the greater, and e = the lesser Index.

Then	1	} according to the <i>Data</i>
And	2	
$2 \div a$	3	
$3 \odot^2$	4	
$1 - 4$	5	
5×81	6	
$6 + 49aa$	7	
$7 \div 130$	8	
$8 \omega^2$	9	
3 and 9	10	

$$\begin{array}{l}
 aa + ee = 520 \\
 ae = \frac{7}{9} aa \\
 e = \frac{7}{9} a \\
 ee = \frac{49}{81} aa \\
 aa = 520 - \frac{49}{81} aa \\
 81aa = 42120 - 49aa \\
 130aa = 42120 \\
 aa = \frac{42120}{130} = 324 \\
 a = \sqrt{324} = 18, \text{ whose Letter is } s. \\
 e = \frac{7}{9} a = 14, \text{ whose Letter is } o.
 \end{array}$$

Hence the Letters will stand thus,

$$Soli. * e * * * * . Gl. * * i * .$$

5. The Difference between the two last Indices, is the Index of the first Letter of the second Word, *viz.* $18 - 14 = 4$ being the Index of the Letter *D*. Then the Letters will stand thus,

$$Soli. De * * * * . Gl. * * i * .$$

6. The

6. The third and last Letter of the second Word, also the third Letter of the third Word, are the same with the second Letter of the first Word; hence the Letters will stand thus,

*Soli Deo Glo * i *.*

7. The Sum of the Indices of the fourth Letter of the third Word, and the sixth or last Letter of the same Word, being added to their Product is 35; and the Difference of their Squares is 288; the Index of the last Letter being the least.

Put a = the greater, and e = the lesser Index, as before.

Then	1	$ae + a + e = 35$	} by the <i>Data</i> .
And	2	$aa - ee = 288$	
$1 - a$	3	$ae + e = 35 - a$	
$3 \div a + 1$	4	$e = \frac{35 - a}{a + 1}$, for $e \times a + 1 = ae + e$	
$4 \odot^2$	5	$ee = \frac{1225 - 70a + aa}{aa + 2a + 1}$	
$2 + 5$	6	$aa = 288 + \frac{1225 - 70a + aa}{aa + 2a + 1}$	
$6 \times aa \&c.$	7	$\begin{cases} a^4 + 2a^3 + aa = 288aa + 576a + 288 \\ + 1225 - 70a + aa \end{cases}$	
$7 +$	8	$a^4 + 2a^3 - 288aa - 506a = 1513$	

This last Equation being resolved according to the Method which shall be shewed in the next Chapter, it will be $a = 17$ it's Letter; and from the 4th Step $e = \frac{35 - a}{a + 1} = 1$, the Index of the Letter a . Then these two Letters being placed according to the *Data* above, are all that are required by the Enigma to compleat these Words

Soli Deo Gloria.

C H A P. X.

The Solution of Adfected Equations in Numbers.

BEFORE we proceed to the Solution of Adfected Equations, it may not be amiss to shew the Investigation (or Invention) of those Theorems or Rules for extracting the Roots of Simple Powers, made use of in Chapter 11. Part 1. I shall here make choice of the same Letters to represent the Numbers both given and sought, as in my Compendium of *Algebra*.

Viz. Let $\begin{cases} G, & \text{always denote the given Resolvend.} \\ r = & \begin{cases} \text{any Number taken as near the true Root as} \\ \text{may be, whether it be greater or less.} \end{cases} \\ e = & \begin{cases} \text{the unknown Part of the Root sought by} \\ \text{which } r \text{ is to be either increased or decreased.} \end{cases} \end{cases}$

Then if r be any Number less than the true Root, it will be $r + e =$ the Root sought. But if r be taken greater than the true Root, it will then be $r - e =$ the Root sought. And put D for the Dividend that is produced from G , after it is lessened and divided by r , &c. (into the Co-efficients of Adfected Equations) according as the Nature of the Root requires. These Things being premised, we may proceed to raising the Theorems.

S E C T. I.

I. **F**OR the Square Root, *viz.* $aa = G$. Quere a .

Let	1	$r + e = a$	
1 $\odot^2$	2	$rr + 2re + ee = aa = G$	
2 $- rr$	3	$2re + ee = G - rr$. Call it D , <i>viz.</i> $D = G - rr$.	
Then	4	$\left\{ \frac{D}{2r + e} = e \right\}$	This shews the 1st Method of extracting the Square Root, Sect. 5. Chap. 11. Part 1.
3 $\div 2$	5	$re + \frac{1}{2} ee = \frac{G - rr}{2} = D$.	

Which gives this Theorem $\left\{ \frac{D}{r + \frac{1}{2} e} = e \right\}$.

The Arithmetical Operations of both these Theorems, you have in the Examples of Section 2. Page 126, to which I refer the

the Learner, supposing him by this Time to understand them without any more Words than what is there expressed.

II. To extract the Cube Root; viz. $aaa = G$. Quære a .

$$\begin{array}{l|l|l} \text{Let } 1 \text{ } \textcircled{2} & 1 & r + e = a, \text{ supposing } r \text{ less than the true Root.} \\ 2 - rrr & 2 & rrr + 3rre + 3ree + eee = aaa = G \\ & 3 & 3rre + 3ree + eee = G - rrr \\ 3 \div 3r & 4 & re + ee + \frac{eee}{3r} = \frac{G - rrr}{3r} = D \end{array}$$

Let $\frac{eee}{3r}$ be rejected or cast off, as being of small Value; then it will be, $re + ee = D$, which gives this following

$$\text{Theorem } \frac{D}{r + e} = e$$

By this Theorem or Rule, the 1st and 2d Examples in Case 1. Page 132, are performed; the which being compared with this Theorem may be easily understood.

Again, Suppose $aaa = G$, as before, and let r be taken greater than the true Root.

$$\begin{array}{l|l|l} \text{Then } 1 \text{ } \textcircled{3} & 1 & r - e = a \\ 2 \pm & 2 & rrr - 3rre + 3ree = a^3 = G \\ 3 \div 3r & 3 & 3rre - 3ree = rrr - G \\ & 4 & re - ee = \frac{rrr - G}{3r} = D \end{array} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} eee \text{ being rejected} \\ \text{as before.} \end{array}$$

Which gives this Theorem $\frac{D}{r - e} = e$.

By this Theorem the third Example in Case 2. Page 133, is performed.

III. To extract the Biquadrate Root; viz. $a^4 = G$. Quære a .

$$\begin{array}{l|l|l} \text{Let } 1 \text{ } \textcircled{4} & 1 & r - e = a \text{ supposing } r \text{ less than just.} \\ 2 - r^4 & 2 & r^4 + 4rrre + 6rree = a^4 = G \\ & 3 & 4rrre + 6rree = G - r^4 \\ 3 \div 2rr & 4 & 2re + 3ee = \frac{G - r^4}{2rr} = e \end{array} \quad \left. \begin{array}{l} \\ \\ \end{array} \right\} \begin{array}{l} \text{rejecting all the} \\ \text{Powers of } e \text{ a-} \\ \text{bove } ee, \end{array}$$

Which gives this Theorem $\frac{D}{2r + 3e} = e$.

By this Theorem the Biquadrate Root of any Number may be extracted. But, as I have already said, Page 134, those Extractions may be very well performed by two Extractions of the Square Root. *Vide Example, Page 135.*

IV. To extract the *Sur-solid* Root, viz. $a^5 = G$. Quære a .

If r be taken less than just, then $r + e = a$, as before, and $\frac{G - r^5}{5r^3} = D$, which gives this Theorem $\frac{D}{r - 2e} = e$. By this Theorem the *Sur-solid* Root, Example 1, Page 136, is extracted. But if r be taken greater than just; then $r - e = a$, and $\frac{r^5 - G}{5r^3} = D$, which gives this Theorem $\frac{D}{r - 2e} = e$. By this last Theorem the Example in Page 137 is performed.

I presume it needless to pursue the raising of those Theorems, for extracting the Roots of Simple Powers, any further; because the Method of doing it is general, how high soever they are; and therefore it may be easily understood by what is already done.

S E C T. 2.

NOTwithstanding I have already shewed the Solution of Quadratick Equations, two several Ways, viz. by casting off the lowest Term; and by completing the Square, *vide* Section 2. Page 195, &c. Yet it may not be amiss to shew, how those Equations may be resolved into Numbers by this Universal Method of continued Series; wherein, if the first r be taken equal to the first true Root, or single Side of the Resolvend; and every single Value of e (as it becomes found) be still added to it, for a new r , then those Roots may be extracted without repeating a second Operation, as before in the single Powers.

Case 1. Let $aa + 2ba = G$. It is required to find the Value of a .

Put	1	$r + e = a$
1 $\odot^2$	2	$rr + 2re + ee = aa$
1 $\times 2b$	3	$2br + 2be = 2ba$
2 + 3	4	$rr + 2br + 2re + 2be + ee = aa + 2ba = G$
4 - rr &c.	5	$2re + 2be + ee = G - rr - 2br$
5 $\div 2$	6	$re + be + \frac{1}{2}ee = \frac{1}{2}G - \frac{1}{2}rr - br = D$

Which gives this Theorem $\frac{D}{r + b + \frac{1}{2}e} = e$.

Suppose

Suppose $b = 364$, and $G = 38692865$: If $r = 6000$, then $rr = 36000000$, and $2br = 4368000$. But $36000000 + 4368000 = 40368000 > 38692865 = G$. Therefore the first $r < 6000$. Let $r = 5000$, then

$ \begin{array}{r} \text{1st } r = 5000 \\ \quad b = 364 \\ \hline \text{1st } r + b = 5364 \\ + \frac{1}{2} e = 400 \\ \hline \text{1 Divisor } 5764) \\ \text{2d } r + b = 6164 \\ + \frac{1}{2} e = 30 \\ \hline \text{2 Divisor } 6194) \\ \text{3d } r + b = 6224 \\ + \frac{1}{2} e = 3.5 \\ \hline \text{3 Divisor } 6227.5 \\ \text{First } r = 5000 \\ + e = 867 \end{array} $	$ \begin{array}{r} 19346432.5 = \frac{1}{2} G \\ - 14320000 = \frac{1}{2} rr + br \\ \hline 5026432.5 = D \quad (800 = e \\ 46112 \\ \hline 41523 \quad (60 = e \\ 37164 \\ \hline 43592.5 \quad (7 = e \\ 43592.5 \quad 867 = e \\ \hline (0) \end{array} $
---	--

$$\left. \begin{array}{l} \text{First } r = 5000 \\ + e = 867 \end{array} \right\} = 5867 = a \text{ as was required.}$$

Case 2. If $aa - 2ba = G$, then proceeding as above, there will arise this *Theorem* $\frac{D}{r - b + \frac{1}{2}e} = e$, &c. And in *Case 3*, viz. $2ba - aa = G$, you will have this *Theorem* $\frac{D}{b - r - \frac{1}{2}e}$ &c. as above.

I think it needless to trouble the Reader with the Work of these two Theorems in Numbers; because if the last Example of *Case 1*, be understood, the other will be easy. Not but that the Method of compleating the Square is very ready and easy, as you may observe by the Work in several Questions of this Chapter.

S E C T. 3.

IN the Solution of all Adfected Equations, that are above (or higher than) Quadratics, it will be the best way to take $r =$ the next nearest Root of the Equation: And then it will be $r + e = a$, if r be less than just; or $r - e = a$ if r be greater than just (as at the Beginning of this Chapter). And all the Powers of the unknown Part of the Root, (viz. e) above it's Square (ee) are to be rejected or cast off, as before in raising the Theorems for the Simple

Simple Powers. And therefore it is, that to supply the want of those Powers (above ee in the Theorem) the Operation must be repeated: as in the Example of extracting the Cube Root, Page 133, viz. when the Figures in the Root consist of more than three Places. (vide Page 140, and 141.)

Suppose $aaa + ba = G$. Quære a .

Let	1	$r + e = a$ viz. let r be supposed less than just.
1 $\odot^3$	2	$rrr + 3rre + 3ree = aaa$
1 $\times b$	3	$br + be = ba$
2 $+ 3$	4	$rrr + br + 3rre + be + 3ree = a^3 + ba = G$
4 $\div 3r$	5	$\frac{1}{3}rr + \frac{1}{3}b + re + \frac{be}{3r} + ee = \frac{G}{3r}$
5 $- \&c.$	6	$re + \frac{be}{3r} + ee = \frac{G}{3r} - \frac{1}{3}rr - \frac{1}{3}b = D$

Which gives this Theorem $\frac{D}{r + \frac{b}{3r} + e} = e$.

But if r be taken greater than just, then it will be $re + \frac{be}{3r} - ee = \frac{1}{3}rr + \frac{1}{3}b - \frac{G}{3r} = D$, which produces this Theorem

$$\frac{D}{r + \frac{b}{3r} - e} = e.$$

By either of these two Theorems the Value of a may be easily found. Or rather otherwise, as in the following Example.

Let $aaa + 24a = 587914$. Here $b = 24$. Suppose the first $r = 90$, then $r^3 = 729000 > 587914$ without the 24×90 being added to it: Therefore $r < 90$. Again, Suppose $r = 80$ then $r^3 = 512000$, and $24r = 1920$. But $512000 + 1920 = 513920 < 587914$, hence > 70 , but nearer to it than 90. Therefore

it must be	1	$r + e = a$ less than just.
1 $\odot^3$	2	$rrr + 3rre + 3ree = aaa$
1 $\times 24$	3	$24r + 24e = 24a$
2 in Numb.	4	$512000 + 19200e + 240ee = aaa$
3 in Numb.	5	$1920 + 24e = 24a$
4 $+ 5$	6	$513920 + 19224e + 240ee = 587914$
5 $- 513920$	7	$19224e + 240ee = 73994$
7 $\div 240$	8	$80,1e + ee = 308,31 = D$
8 $\div$	9	$e = \frac{D}{80,1 + e}$

Operation

Operation 80,1

$$+e = 3,$$

1 Divisor 83,1)

$$+e = 3,6$$

2 Divisor 80,7)

$$+e = ,67$$

$$87,37)$$

$$\begin{array}{r} 308,31 \quad (80, = r \\ 249,3 \quad \underline{3,68 \text{ \&c.} = e} \\ 59,01 \quad 83,68 \text{ \&c.} = r + e \\ 52,02 \\ 6,99 \text{ \&c.} \end{array}$$

Or rather new $r=83,7$ for a second Operation, which being involved and tried (as above) will be found greater than just: therefore

it must be	1	$r - e = a$
1 $\odot^3$	2	$rrr - 3rre + 3ree = aaa$
1 $\times 24$	3	$24r - 24e = 24a$
2 in Numb.	4	$586376,253 - 21017,07e + 251,1ee = aaa$
3 in Numb.	5	$2008,8 - 24e = 24a$
4 + 5	6	$588385,053 - 21041,07e + 251,1ee = 587914$
6 +	7	$21041,07e - 251,1ee = 471,053$
7 $\div 251,1$	8	$83,7955e - ee = 1,87595778 = D$
8 $\div$	9	$e = \frac{D}{83,7955 - e}$

2d Operation 83,7955

$$-e = ,02$$

1st Divisor 83,7755)

$$-e = ,022$$

2d Divisor 83,7535)

$$-e = ,0023$$

3d Divisor 83,7512)

$$-e = 3 \text{ \&c.}$$

$$83,751$$

$$\begin{array}{r} 1,87595778 \quad (83,70000000 = r \\ 1,675510 \quad \underline{00,02239331 = e} \\ ,2004477 \quad 83,67760669 = a = r \\ ,1675070 \quad - e \\ ,03294078 \\ ,02512536 \\ ,00781542 \\ 00753760 \end{array}$$

Here the new Divisors are rejected, as insignificant.

$$\begin{array}{r} 27782 \\ 25125 \\ 2657 \\ 2512 \\ 145 \\ 83 \end{array}$$

All the remaining Examples of extracting Roots (except Page 260) are left in the Author's own Method; which by this Time, it is presumed, the Learner will easily know how to correct of himself, if he takes due Notice of what has been delivered Page 131, 132, &c.

But

But if more Exactness be required, you may make the new $r = 83,677,6067$, and proceed with it to a third Operation; which will afford twenty-seven Places of Figures for the Value of a ; that is, every Operation will produce triple the Places of Figures to those of the Precedent r . And this tripling the Places of Figures in the Root, at every Operation, holds good, and is to be observed in the Solution of all Adfected Equations (how high soever they are) according to this Method of resolving them. See Page 141.

Example 2. Suppose $aaa - ba = G$. Quære a . If $r + e = a$, then $re - \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$, which gives this Theorem $\frac{D}{r - \frac{\frac{1}{3}b}{r} + e} = e$. But if $r - e = a$, then $re + \frac{\frac{1}{3}be}{r} + ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}b - \frac{1}{3}rr = D$, which gives this Theorem $\frac{D}{r + \frac{\frac{1}{3}b}{r} + e} = e$.

Or you may proceed otherwise, as in the last Example. Let $aaa - 6438a = 104785688$, here $b = 6438$. Suppose the first $r = 500$, $rrr = 125000000$, and $br = 3219000$, then $125000000 - 3219000 = 121781000$. But $121781000 > 104785688$, therefore $r < 500$. Again, suppose $r = 400$, $rrr = 64000000$, and $br = 2575200$, then will $64000000 - 2575200 = 6142800$. But $6142800 < 104785688$, hence $r > 400$; consequently r is betwixt 400 and 500. But 500 is the next nearest; therefore, let $r = 500$ being greater than just.

Then	1	$r - e = a$
$1 \odot^2$	2	$rrr - 3rre + 3ree = aaa$
$1 \times b$	3	$br - be = ba$
2 in Numb.	4	$125000000 - 750000e + 1500ee = aaa$
3 in Numb.	5	$3219000 - 6438e = 6438a$
4 - 5	6	$121781000 - 743562e + 1500ee = 104785688$
6 +	7	$743562 - 1500ee = 16995312$
$7 \div 1500$	8	$495e - ee = 11330 = D$
8 ÷	9	$e = \frac{D}{495 - e}$

Operation

Operation 495

$$-e = 20$$

1 Divisor 475)

$$-e = 3$$

$$472)$$

$$\begin{array}{r} 11330 \quad (500,0=r \\ 23,8=e \\ 950 \quad 476,2=r-e=a \\ 1830 \\ 1416 \\ \hline 414,0 \\ 377,6 \end{array}$$

Let new $r=476$ for a 2d Operation, then $r^3=107850176$ and $br=3064488$: but $107850176-3064488=104785688$ the same with the Resolvend. Consequently $a=476$ just.

Example 3. Let $ba-aaa=G$. Quære a . If $r+e=a$, then $\frac{\frac{1}{3}be}{r} - re - ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$, which gives this

Theorem $\frac{D}{\frac{\frac{1}{3}b}{r} - re} = e$. But if $r-e=a$, then $re - \frac{\frac{1}{3}be}{r}$

$-ee = \frac{\frac{1}{3}G}{r} + \frac{1}{3}rr - \frac{1}{3}b = D$, which gives this *Theorem*

$$\frac{D}{r - \frac{\frac{1}{3}b}{r} - e} = e.$$

Or otherwise as before in the two last Examples. Thus, let $123456a-aaa=12272861$. Here $b=123456$. Suppose the first $r=200$, then $rrr=8000000$, and $br=24691200$; then $24691200-8000000=16691200$, but $16691200 > 12272861$, therefore r is here less than just, because the highest Power is —, or Negative. Again, Suppose $r=300$, then $r^3=27000000$, and $br=37036800$, then $37036800-27000000=10036800 < 12272861$. Consequently $r < 300$, and $r > 200$. Let $r=300$, being the next nearest, but more than just.

Then	1	$r - e = a$
1 $\odot^3$	2	$rrr - 3rre + 3ree = aaa$
1 $\times b$	3	$br - be = ba$
2 in Numb.	4	$27000000 - 270000e + 900ee$
3 in Numb.	5	$37036800 - 123456e$
5 — 4	6	$10036800 + 146544e - 900ee = 12272861$
6 —	7	$146544e - 900ee = 2236061$
7 $\div 900$	8	$162e - ee = 2484 = D$
1 $\div$ &c.	9	$e = \frac{D}{162 - e}$

I i

[Operation

$$\begin{array}{rcl}
 \text{Operation} & 162 & \\
 -e = & 10 & \\
 \hline
 \text{1st Divisor} & 152) & \\
 -e = & 6 & \\
 \hline
 \text{2d Divisor} & 646) &
 \end{array}$$

$$\begin{array}{rcl}
 2484 & (300,0 = r & \\
 152 & 16,6 = e & \\
 \hline
 & 283,4 = r - e = a. & \\
 964 & & \\
 876 & & \\
 \hline
 & 88,0 & \\
 & 86,6 & \\
 \hline
 & &
 \end{array}$$

Or new $r \pm 283$, which being involved, &c. will appear to be the true Root, that is, $a = 283$ just.

Note, These are usually called the three Forms of Cubick Equations; and in the Solution of the third or last Form, viz. $ba - aaa = G$, you may meet with some seeming Difficulties; especially in making Choice of the first r , because this Equation is an ambiguous Equation, and hath two Affirmative Roots, viz. a greater and lesser Root. But having once found either of them, the other may be easily obtained by Division only; as in the Quadratick Equations. *Vide* Chap. 8. As for Instance, in the last Example, $a = 283$ and $123456a - aaa = 12272861$. Make these two Equations $= 0$, to wit, let $a - 283 = 0$, and $-aaa + 123456a - 12272861 = 0$.

$$\begin{array}{rcl}
 \text{Then, } a - 283) & -aaa + 123456a - 12272861 & (-aa \\
 & -aaa + 283aa & \\
 \hline
 & -283aa + 123456a & (-283a \\
 & -283aa + 80689a & \\
 \hline
 & +43367a - 12272861 & (+43367 \\
 & +43367a - 12272861 & \\
 \hline
 & (0) & (0)
 \end{array}$$

Hence it appears that $-aa - 283a + 43367 = 0$. Consequently $aa + 283a = 43367$ this Equation being solved, $a = 110$, 2722 &c. which is the lesser Root of the aforesaid Equation $ba - aaa = G$, &c. After this Manner all the possible and impossible Roots of any Equation may be easily discovered, any one of it's Roots being once found. I shall therefore omit inserting more Examples of that kind.

Suppose $aaa + baa + ca = G$. Quære a . Let $b = 74$, $c = 8729$, and $G = 560783$. By Trial (as before) it will be found that the next nearest $r = 40$ being something less than just.

Therefore

Therefore	1	$r + e = a$
$1 \times e$	2	$cr + ce = ca$
$1 \otimes^2 : \times b$	3	$brr + 2bre + bee = baa$
$1 \otimes^3$	4	$rrr + 3rre + 3ree = aaa$
2 in Numb.	5	$349160 + 8729e$
3 in Numb.	6	$118400 + 5920e + 74ee$
4 in Numb.	7	$64000 + 4800e + 120ee$
5 + 6 + 7	8	$531560 + 19449e + 194ee = 560783$
8 - 531560	9	$19449e + 194ee = 29223$
$9 \div 194$	10	$100,2e + ee = 153,06 = D$
$10 \div$	11	$e = \frac{D}{100,2 + e}$

Operation	100,2	
$+ e =$	$\cdot 1$	
1st Divisor	101,2)	$153,06$ ($40,0 = r$)
$+ e =$	$\cdot 5$	$101,2$ ($1,5 = e$)
2d Divisor	101,7)	$41,5 = r + e = a$
		$51,86$
		$50,85$
		$1,01$

Or new $r = 41,5$ for a second Operation, which being duly involved, &c. will be found more than just.

Therefore	1	$r - e = a$
Then	2	$cr - ce = ca$
	3	$brr - 2bre + bee = baa$
	4	$rrr - 3rre + 3ree = aaa$

These being turned into Numbers, &c. as above, they will be $20037,75e - 198,5ee = 390,375$, which being divided by 198,5 the Co-efficient of ee , will become $100,946e - ee = 1,956624$, &c. = D .

Operation	100,946	
$- e =$	01	
1st Divisor	100,936)	$1,966624$ ($41,5000000 = r$)
$- e =$	$\cdot 009$	$\cdot 0194847 = e$
2d Divisor	100,927)	$41,4805153 = r - e = a$
		957264
		908343

* Here I proceed by plain Division without forming new Divisors.

*	489210
	<u>403708</u>
	855020
	<u>807416</u>
	476040
	<u>403708</u>
	72332 &c.

I i 2

Let

Let the last Equation in the Enigma, Chap. 9. be here proposed for a Solution. *Viz.* $aaaa + baaa - caa - da = G$; $b = 2$, $c = 288$, $d = 506$, and $G = 1513$, Quære a . By Trials it will be found, that the next nearest $r = 20$, being something more than just.

Therefore	1	$r - e = a$
$1 \times d$	2	$dr - de = da$
$1 \text{ @ }^2 \times c$	3	$crr - 2cre + cee = caa$
$1 \text{ @ }^3 \times b$	4	$brrr - 3brre + 3bree = baaa$
1 @ ^4	5	$r^4 - 4rrre + 6rree = aaaa$

These being turned into Numbers, and those duly collected, according as the Signs of the Equation direct, they will become $50680 - 22374e + 2232ee = 1513$, which being all divided by 2232 the Co-efficient of ee , will be $10e - ee = 22 = D$.

$$\text{Then } \frac{D}{10 - e} = e.$$

Operation	10
$- e =$	3
Divisor	7)

22	(	$20 = r$
21		$3 = e$
1		$17 = r - e = a$ just.

See the End of Chap. 9.

By what hath been already done about the Solution of these few Equations (being carefully observed) I presume the Learner will easily conceive how to proceed in the Solution of all Kinds of Equations, be they never so high, or adfected; therefore I shall not here propose many various Examples, but only take them as they fall in Course, when I come to the next Part, wherein you will (perhaps) find such Equations with their Solutions as are not common.

C H A P. XI.

Of Simple Interest, Annuities, or Pensions, &c.

INTEREST, or the Use paid for the Loan of Money, is either Simple, or Compound.

Sect. 3. *Of Simple Interest.*

SIMPLE Interest, is that which is paid for the Loan of any Principal or Sum of Money, lent out for some Time, at any Rate *per Cent.* agreed on between the Borrower and the Lender; which, according to the late Laws of *England*, ought to be six Pounds for the Use of 100 *l.* for one Year, and twelve Pounds for the Use of 100 *l.* for two Years; and so on for a greater, or lesser Sum, proportionable to the Time proposed.

There are several Ways of computing (or answering Questions about) Simple Interest; as by the single and double Rule of Three (See Page 96, &c.) others make use of Tables composed at several Rates *per Cent.* as Sir *Samuel Moreland*, in his *Doctrine of Interest*, both simple and compound, all performed by Tables; wherein he hath detected several material Errors committed by Sir *Isaac Newton*, Mr *Kersey* upon *Wingate*, and Mr *Clavil*, &c. in the Business of computing Interest, &c. by their Tables, too tedious to be here repeated. But I shall in this Tract take other Methods, and shew that all Computations relating to Simple Interest are grounded upon Arithmetick Progression; and from thence raise such general Theorems, as will suit with all Cases. In order to that

Let $\begin{cases} P = \text{any Principal or Sum put to Interest.} \\ R = \text{the Ratio of the Rate, per Cent. per Annum.} \\ t = \text{the Time of the Principal's Continuance at Interest.} \\ A = \text{the Amount of the Principal, and it's Interest.} \end{cases}$

Note, The Ratio of the Rate, is only the Simple Interest of 1 *l.* for one Year, at any given Rate; and is thus found.

Viz. $100 : 6 :: 1 : 0,06$ = the Ratio at 6 *per Cent. per Annum.*

Or $100 : 7 :: 1 : 0,07$ = the Ratio at 7 *per Cent. &c.*

Again $100 : 7,5 :: 1 : 0,075$ = the Ratio at 7 and $\frac{1}{2}$ *per Cent.*

And if the given Time be whole Years; then t = the Number of whole Years: but if the Time given, be either pure Parts of a Year, or Parts of a Year mixed with Years; those Parts must be turned into Decimals; and then t = those Decimals, &c.

Now

Now the common Parts of a Year may be easily turned or converted into Decimal Parts, if it be considered

That one $\left\{ \begin{array}{l} \text{Day is the } \frac{1}{365} \text{ Part of a Year} = 0,00274 \text{ ferè} \\ \text{Month is the } \frac{1}{12} \text{ Part of a Year} = 0,0833333 \text{ \&c.} \\ \text{Quarter is the } \frac{1}{4} \text{ Part of a Year} = 0,25 \end{array} \right.$

These Things being premised, we may proceed to raising the *Theorems*.

Let R = the Interest of 1*l*. for one Year, as before.

Then 2 R = the Interest of 1*l*. for two Years.

And 3 R = the Interest of 1*l*. for three Years.

4 R = the Interest of 1*l*. for four Years. And so on for any Number of Years proposed.

Hence it is plain, that the Simple Interest of one Pound is a Series of Terms in Arithmetic Progression increasing; whose first Term and common Difference is R , and the Number of all the Terms is t . Therefore the last Term will always be tR = the Interest of 1*l*. for any given Term signified by t .

Then $\left\{ \begin{array}{l} \text{As one Pound : is to the Interest of 1*l*. :: so is any} \\ \text{Principal or given Sum : to it's Interest.} \end{array} \right.$

That is, $1*l*. : tR :: P : tRP$ = the Interest of P . Then the Principal being added to it's Interest, their Sum will be = A the Amount required; which gives this general *Theorem*.

Theorem 1. $tRP + P = A$.

From whence the three following *Theorems* are easily deduced.

Theorem 2. $\frac{A}{tR + 1} = P$. *Theorem 3.* $\frac{A - P}{tP} = R$.

Theorem 4. $\frac{A - P}{RP} = t$.

These four *Theorems* resolve all Questions about Simple Interest.

Question 1. What will 256*l*. 10*s*. amount to in 3 Years, one Quarter, 2 Months, and 18 Days, at 6 per Cent. per Annum.

Here is given $P = 256,5$; $R = 0,06$; and $t = 3,46599$.

For 3 Years = 3

Quære A . per *Theorem 1*.

one Quarter = 0,25

2 Months = 0,16667 = $0,08333 \times 2$

18 Days = 0,04932 = $0,00274 \times 18$

Hence $t = 3,46599 : \times 0,06 = 0,2079594 = tR$

Then $0,2079594 \times 256,5 = 53,341586 = tRP$

And $53,341586 + 256,5 = 309,841586 = tRP + P = A$.

That is, 309,841586 = 309*l*. 16*s*. 10*d*. being the Answer required.

Question

Question 2. What Principal or Sum being put to Interest, will raise a Stock of 309 l. 16 s. 10 d. in three Years, one Quarter, two Months, and 18 Days; at 6 per Cent. per Annum?

Or the same Question otherwise stated thus.

What is 309 l. 16 s. 10 d. due 3 Years, one Quarter, 2 Months and 18 Days hence, worth in ready Money; abating or discounting 6 per Cent. &c.

Here is given $A = 309,841586$; $R = 0,06$; $t = 3,46599$ (found as before) thence to find P . Per Theorem 2. First $3,46599 \times 0,06 = 0,2079594 = tR$. Then

$tR + 1 = 1,2079594$ $309,841586 = A$ ($256,5 = P$;
that is, 256 l. 10 s. the Answer required.

Question 3. At what Rate or Interest, per Cent. &c. will 256 l. 10 s. amount to 309 l. 16 s. 10 d. in three Years, one Quarter, two Months, and 18 Days?

Here is given, $P = 256,5$; $A = 309,841586$; and $t = 3,46599$ to find R . Per Theorem 3. First $309,841586 - 256,5 = 53,341586 = A - P$. Next $3,46599 \times 256,5 = 889,026435 = tR$. And $tR = 889,026435$ $53,341586$ ($0,06 =$ the Ratio. Then $1l. : 0,06 :: 120 : 6 =$ the Rate required.

Question 4. In what Time will 256 l. 10 s. raise a Stock of (or amount to) 309 l. 16 s. 10 d. at 6 per Cent. &c.

Here is given, $P = 256,5$; $A = 309,841586$, and $R = 0,06$ to find t . Per Theorem 4. First $309,841586 - 256,5 = 53,341586 = A - P$. And $256,5 \times 0,06 = 15,39 = PR$. Then $15,39$ $53,341586$ ($3,46599 = t$; that is $t = 3$ Years and ,46599 Decimal Parts of a Year; which may be brought into common Parts of a Year, thus

$$\begin{array}{r} 0,46599 \\ 0,25 = \text{one Quarter.} \\ \hline 0,21599 \end{array} \quad \begin{array}{r} \text{And } 0,08333) 0,21599 \text{ (2 Months.} \\ \underline{,16666} \\ 0,02074) ,04933 \text{ (18 Days.} \end{array}$$

Hence $t = 3$ Years, one Quarter, 2 Months, and 18 Days; the Answer required.

It must needs be easy to conceive, that what is here done at 6 per Cent. may be done at any other Rate of Interest, by forming the Ratio (*viz.* R) accordingly.

S C H O L I U M.

Although it be according to the Laws and Custom of *England*, to compute Interest at the Proportion of 6 *per Cent.* (as above) yet he that takes up Money at Interest for any Time less than even or compleat Years, pays more Interest than seems reasonably due, according to the Rules of Art. As for Instance; if 100 *l.* be forborne at Interest one whole Year, it amounts to 106 *l.* But (I say) if it be paid at the half Year's End, it should not amount to 103; as appears from this following Proportion.

Let a = the Amount due at the half Year's End; then it will be $100 : a :: a : 106$ the Amount at the Year's End. *Ergo* $aa = 10600$, and $a = \sqrt{10600} = 102,9563 = 102\text{ l. } 19\text{ s. } 1\frac{1}{2}\text{ d.}$ which is less than 103 *l.* by $10\frac{1}{2}\text{ d.}$ And if it be paid in less than half a Year's Time, the Error must needs be the greater.

Sect. 2. Of Annuities, or Pensions in Arrears, computed at Simple Interest.

ANNUITIES, or Pensions, &c. are said to be in Arrears, when they are payable or due, either Yearly, or Half-yearly, &c. and are unpaid for any Number of Payments. Therefore the Business is, to compute what all those Payments will amount unto, allowing any Rate of Simple Interest for their Forbearance, from the Time each particular Payment became due: Now in order to that,

Put $\begin{cases} u = \text{the Annuity, Pension, or Yearly Rent, \&c.} \\ t = \text{the Time of it's Continuance, or being unpaid.} \\ R = \text{the Ratio, or Interest of 1 l. for 1 Year, as before.} \\ A = \text{the Amount of the Annuity and it's Interest.} \end{cases}$

Then if u = the first Year's Rent, due without Interest.

$\begin{matrix} Ru = \text{the Interest} \\ 2u = \text{the Rent} \end{matrix} \left. \vphantom{\begin{matrix} Ru \\ 2u \end{matrix}} \right\} \text{ due at the End of the second Year.}$
 $\begin{matrix} 2Ru = \text{the Interest} \\ 3u = \text{the Rent} \end{matrix} \left. \vphantom{\begin{matrix} 2Ru \\ 3u \end{matrix}} \right\} \text{ due at the End of the third Year.}$
 $\begin{matrix} 3Ru = \text{the Interest} \\ 4u = \text{the Rent} \end{matrix} \left. \vphantom{\begin{matrix} 3Ru \\ 4u \end{matrix}} \right\} \text{ due at the End of the fourth Year.}$
 $\begin{matrix} 4Ru = \text{the Interest} \\ 5u = \text{the Rent} \end{matrix} \left. \vphantom{\begin{matrix} 4Ru \\ 5u \end{matrix}} \right\} \text{ due at the End of the fifth Year.}$

And so on for any Number of Years. Hence it is evident, that $Ru + 2Ru + 3Ru + 4Ru + 5u = A$ the Sum of all the Rents and their Interest, being forborne 5 Years.

From

From whence it follows, that $Ru + 2Ru + 3Ru + 4Ru = A - tu$.
Here $t = 5$. Divide by u , then $R + 2R + 3R + 4R = \frac{A - tu}{u}$.

Next to find the Sum of this Progression (See Page 185) thus,
Let $R + 2R + 3R + 4R \&c. = z$, then $1 + 2 + 3 + 4 \&c. = \frac{z}{R}$.

Here the Sum of the first and last Terms are $4 + 1 = 5 = t$,
and the Numbers of all the Terms is $4 = t - 1$. Therefore
 $\frac{t-1}{2} \times t =$ the Sum of all the Terms; that is, $\frac{tt-t}{2} = \frac{z}{R}$:

hence $\frac{ttR - tR}{2} = z$. Consequently $\frac{ttR - tR}{2} = \frac{-tu}{u}$.

Now from this Equation it will be easy to deduce the following
Theorems.

Theorem 1. $\frac{ttRu - tRu + 2tu}{2} = A$, or $\frac{ttu - tu}{2} = R : + tu = A$.

Theorem 2. $\frac{2A}{ttR - tR + 2t} = u$. Theorem 3. $\frac{2A - 2tu}{ttu - tu} = R :$

Let $\frac{2}{R} - 1 = x$, then $t = \sqrt{\frac{2A}{Ru} + \frac{xx}{4}} : - \frac{1}{2} \times$ Theorem 4.

Question 1. If 250 l. yearly Rent (or Pension, &c.) be forborn
or unpaid seven Years; what will it amount to in that Time, at 6
per Cent. for each Payment, as it becomes due?

Here is given $u = 250$, $t = 7$, and $R = 0,06$; to find A . Per
Th. 1. First $250 \times 7 = 1750 = tu$, $1750 \times 7 = 12250 = ttu$.
Again $12250 - 1750 = 10500 = ttu - tu$, and $\frac{10500}{2} \times 0,06 = 315$:
Lastly $315 + 1750 = 2065 = A$; Viz. 2065 l. is the Answ. required.

But if the Annuity, Rent, or Pension, is to be paid by Quarter-
ly or half yearly Payments, &c. Then $\frac{0,06}{2} = 0,03 = R$ for
half yearly Payments: and $\frac{0,06}{4} = 0,015 = R$ for quarterly;
or $0,045 = R$ for three quarterly Payments. Example of half
yearly Payments.

Suppose 250 l. per Annum, to be paid by half yearly Payments,
were in Arrears, or unpaid for seven Years; what would it amount
to, allowing 6 per Cent. per Annum for each Payment, as it be-
comes due?

In this Example there is given $u = 125 = \frac{250}{2}$; $t = 14$ the Number of Payments; and $R = 0,03 = \frac{0,06}{2}$; thence to find A .

First $125 \times 14 = 1750 = tu$; $1750 \times 14 = 24500 = ttu$; again $24500 - 1750 = 22750 = ttu - tu$; then $\frac{22750}{2} = 11375$, and $11375 \times 0,03 = 341,25$. Lastly $341,25 + 1750 = 2091,25$; that is, $A = 2091\text{ l. } 5\text{ s.}$ the Answer required.

N. B. Hence it may be observed, that half yearly Payments are more advantageous than yearly. For $2091\text{ l. } 5\text{ s.} > 2065\text{ l.}$ by $26\text{ l. } 5\text{ s.}$ consequently, quarterly Payments are more advantageous than half yearly Payments.

Question 2. What yearly Rent, Pension, &c. being forborn or unpaid seven Years, will raise a Stock of 2065 l. allowing 6 per Cent. per Annum for each Payment, as it becomes due?

Here is given $A = 2065$, $t = 7$, and $R = 0,06$; to find u . Per Theorem 2. First $7 \times 0,06 = 0,42 = tR$, and $0,42 \times 7 = 2,94 = ttR$. Then $ttR - tR = 2,52$. Lastly $ttR - tR + 2t = 16,52$ $4130 = 2A$ ($250 = u$; that is, 250 l. per Annum, &c. will raise 2065 l. the Stock required.

Question 3. In what Time will 250 l. yearly Rent raise a Stock of 2065 l. allowing 6 per Cent. &c. for the Forbearance of the Payments as they become due?

Here is given $u = 250$, $A = 2065$, and $R = 0,06$; to find t . Per Theorem 4. First $\frac{2}{R} = \frac{0,06}{2} = 33,3333$; and $33,3333 - 1 = 32,3333 = x = \frac{2}{R} - 1$. Then $16,16666 \text{ \&c.} = \frac{1}{2}x$; $261,3605 \text{ \&c.} = \frac{1}{4}xx$. Again $\frac{4130}{15} = 275,333 = 2A \div Ru$, and $275,3333 + 261,3605 = 536,6938 = \frac{2A}{Ru} + \frac{1}{4}xx$. Then $\sqrt{536,6938} = 23,1666$. Lastly, $23,1666 - 16,1666 = 7 = t$ the Time required.

Question 4. If 250 l. yearly Rent, being forborn seven Years, will amount to 2065 l. allowing Simple Interest for every Payment as it becomes due; what must the Rate of the Interest be per Cent. &c.?

Here is given $u = 250$, $A = 2065$, and $t = 7$; to find R : Per Theorem 3.

Thus $\begin{cases} ttu = 12250 \\ tu = 1750 \end{cases} \begin{cases} 4130 = 2A \\ 3500 = 2tu \end{cases}$
 $ttu - tu = 10500$ $630 = 2A - 2tu$ ($0,06 = R$).
 Then $1 : 0,06 :: 100 : 6$ the Rate required.

Sect.

Sect. 3. *The Present Worth of Annuities or Pensions, &c. computed at Simple Interest.*

THE Business of purchasing Annuities, or taking of Leases, &c. for any assigned Time, depends upon the true equating of the Principal or Money laid out on the Purchase, with the Annuity or Yearly Rent, by allowing (or discounting) the same Rate of Interest to both Parties. Which may be easily performed by duly applying the respective *Theorems* of the two last Sections together; as will fully appear by the following Question.

Question 1. What is 75 l. yearly Rent, to continue nine Years, worth in ready Money, at 6 per Cent. per Annum Simple Interest?

1. Per *Theorem 1.* of the last Section, find what the proposed yearly Rent would amount to, if it were forborn 9 Years, at 6 per Cent.

Thus $u = 75$, $t = 9$, and $R = 0,06$:	Quære A .
$ttu = 6075$	Then 2) 5400 (2700)
$tu = 675$	$R = 0,06$ } Multiply
$ttu - tu = 5400$	$\frac{162,}{+ tu = 675,} = 837 = A$.

2. Then by *Theorem 2. Section 1.* find what Principal, being put to Interest for the same Time, and at the same Rate, will amount to $837 l. = A$. Thus $tR = 0,54 = 9 \times 0,06$; $tR + 1 = 1,54$ $837 (543,5064 = P$: that is, $P = 543 l. 10 s. 1 \frac{1}{2} d$. which is the Worth of 75 l. a Year, as was required.

From the Work of these two Operations (duly considered) it must needs be easy to conceive, how the two *Theorems* by which they were performed, may be combined in one.

For 1. $\frac{ttRu - tRu + 2tu}{2} = A$; and 2. $PtR + P = A$.

Consequently $PtR + P = \frac{ttRu - tRu + 2tu}{2}$. And from this Equation may be deduced the following *Theorems*.

Theorem 1. $\frac{ttRu - tRu + 2tu}{2tR + 2} = P$, or $\frac{ttR - tR + 2t}{2tR + 2} \times u = P$.

By this *Theorem* all Questions of the same Kind with the last (*viz.* that above) may be easily and readily answered at one Operation.

Theorem 2. $\frac{2PtR + 2P}{ttR - tR + 2t} = u$, or $\frac{tR + 1}{ttR - tR + 2t} : \times 2P = u$.

Theorem 3. $\frac{2P - 2tu}{ttu - tu - 2Pt} = R$.

Let $\frac{2}{R} - \frac{2P}{u} - 1 = x$, then will $tt \pm xt = \frac{2P}{Ru}$.

Which gives this *Theorem 4.* $\sqrt{\frac{2P}{Ru} + \frac{xx}{u}} : \pm \frac{x}{2} = t$.

By the second and fourth *Theorems*, two very useful Questions may be easily answered.

1. *As for Instance: If it be required to find what Annuity, or yearly Rent, &c. may be purchased, for any proposed Sum, to continue any assigned Time, allowing any Rate of Interest?*

This Question may be answered by *Theorem 2*.

2. *Again: If it be required to find how long any yearly Rent, Pension, or Annuity, &c. may be purchased (or enjoyed) for any proposed Sum, at any given Rate of Interest?*

All Questions of this Kind are easily answered by *Theorem 4*.

In these Questions it is supposed, that the Purchase, or yearly Rent, is to commence or be immediately entered upon. But if it be required to find the Value or Purchase of an Annuity or yearly Rent, &c. in Reversion; that is, when it is not to be entered upon until after some Time, or Number of Years are past; then you must first find what the Sum proposed to be laid out in the Purchase, would amount to, if it were put to Interest, during the Time the Annuity, &c. is not to be put in present Possession; and make that Amount the Sum for the Purchase, proceeding with it as in either of the two last Questions, &c.

Note, *From the first Question of this Section it will be easy to conceive how to perform the Equation of Payments, between Debtor or Creditor, at any Rate of Interest, without doing any Damage to either Party.*

That is, when several Sums of Money are to be paid, at several different Times, to find the Time when all the Payments may be truly discharged at once: as if one Sum were to be paid at the End of two Months, another at six Months, and perhaps a third Sum at eight Months end, &c. And if it were required to find the Time when all those Sums may be truly discharged at one Payment without Loss, &c.

C H A P. XII.

Of Compound Interest, and Annuities, &c.

COMPOUND Interest is that which arises from any Principal and it's Interest put together, as the Interest so becomes due; so that at every Payment, or at the Time when the Payments became due, there is created a new Principal; and for that Reason it is called Interest upon Interest, or Compound Interest.

As for Instance; Suppose 100 *l.* were lent out for two Years, at 6 *per Cent. per Annum*, Compound Interest: then at the End of the first Year, it will only amount to 106 *l.* as in Simple Interest. But for the second Year this 106 *l.* becomes Principal, which will amount to 112 *l.* 7*s.* 2 $\frac{1}{2}$ *d.* at the second Year's End, whereas by Simple Interest it would have amounted to but 112 *l.*

And altho' it be not lawful to let out Money at Compound Interest; yet in purchasing of Annuities or Pensions, &c. and taking Leases in Reversion, it is very usual to allow Compound Interest to the Purchaser for his ready Money; and therefore it is very requisite to understand it.

Sect. 1. Of Compound Interest.

Let $\left\{ \begin{array}{l} P = \text{the Principal put to Interest.} \\ t = \text{the Time of it's Continuance.} \\ A = \text{the Amount of the Principal and Interest.} \\ R = \left\{ \begin{array}{l} \text{the Amount of 1 } l. \text{ and it's Interest for 1 Year, at} \\ \text{any given Rate, which may be thus found.} \end{array} \right. \end{array} \right\} \text{as before.}$

Viz. 100 : 106 :: 1 : 1,06 = the Amount of 1 *l.* at 6 *per Cent.*

Or 100 : 105 :: 1 : 1,05 = the Amount of 1 *l.* at 5 *per Cent.*

and so on for any other assigned Rate of Interest.

Then if R = the Amount of 1 *l.* for one Year, at any Rate.

R^2 = the Amount of 1 *l.* for two Years.

R^3 = the Amount of 1 *l.* for three Years.

R^4 = the Amount of 1 *l.* for four Years.

R^5 = the Amount of 1 *l.* for five Years. Here $t = 5$

For 1 : R :: R : RR :: RR : RRR :: RRR : R^4 :: R^4 : R^5 : &c. in $\div$.

That is $\left\{ \begin{array}{l} \text{As one Pound : is to the Amount of one Pound at one} \\ \text{Year's End : : so is that Amount : to the Amount of} \\ \text{one Pound at two Years End, \&c.} \end{array} \right.$

Whence

Whence it is plain, that Compound Interest is grounded upon a Series of Terms, increasing in Geometrical Proportion continued; wherein t (*viz.* the Number of Years) does always assign the Index of the last and highest Term: *Viz.* the Power of R , which is R^t .

Again, As $1 : R^t :: P : PR^t = A$ the Amount of P for the Time that $R^t =$ the Amount of $1l$.

That is $\left\{ \begin{array}{l} \text{As one Pound : is to the Amount of one Pound for any} \\ \text{given Time :: so is any proposed Principal (or Sum) to} \\ \text{it's Amount for the same Time.} \end{array} \right.$

From the Premises (I presume) the Reason of the following *Theorems*, may be very easily understood.

Theorem 1. $PR^t = A$, as above.

From hence the two following *Theorems* are easily deduced.

Theorem 2. $\frac{A}{R^t} = P$. *Theorem 3.* $\frac{A}{P} = R^t$.

By these three *Theorems*, all Questions about Compound Interest may be truly resolved by the Pen only, *viz.* without Tables; tho' not so readily as by the Help of Tables, calculated on purpose; as will appear farther on.

Question 1. What will 256l. 10s. amount to in seven Years, at 6 per Cent. per Annum, Compound Interest?

Here is given $P = 256,5$; $t = 7$; and $R = 1,06$ which being involved until it's Index $= t$ (*viz.* 7.) will become $R^7 = 1,50363$. Then $1,50363 \times 256,5 = 385,6811 = A = 385l. 13s. 7\frac{1}{2}d.$ which is the Answer required.

Question 2. What Principal or Sum of Money must be put (or let) out to raise a Stock of 385l. 13s. 7 $\frac{1}{2}$ d. in seven Years, at 6 per Cent. per Annum, Compound Interest?

Here is given $A = 385,6811$; $R = 1,06$; and $t = 7$; to find P , by *Theorem 2*. Thus $R^t = 1,50363$ $385,6811 = A$ ($256,5 = P$). That is, $P = 256l. 10s.$ which is the Principal or Sum, as was required.

Question

Question 3. In what Time will 256 l. 10 s. raise a Stock of (or amount to) 385 l. 13 s. 7 1/2 d. allowing 6 per Cent. per Annum, Compound Interest?

Here is given $P = 256,5$; $A = 385,6811$; $R = 1,06$; to find t by the third Theorem $R^t = \frac{A}{P} = \frac{385,6811}{256,5} = 1,50363$, which being continually divided by $R = 1,06$ until nothing remain, the Number of those Divisions will be $7 = t$. Thus $1,06) 1,50363 (1,41852$. And $1,06) 1,41852 (1,338225$. Again $1,06) 1,338225 (1,262477$. And so on until it become $1,06) 1,06 (1$. which will be at the seventh Division. Therefore it will be $t = 7$ the Number of Years required by the Question.

Question 4. If 256 l. 10 s. will amount to (or raise a Stock of) 385 l. 13 s. 7 1/2 d. in seven Years Time; what must the Rate of Interest be, per Cent. per Annum?

Here is given $P = 256,5$; $A = 385,6811$, and $t = 7$, Quære R . By Theorem 3. $\frac{A}{P} = R^t = 1,50363$; as before in the last Question. And if $R^t = R^7 = 1,50363$, then $R = \sqrt[7]{1,50363}$, which may be thus extracted.

Put	1	$r + e = R$, then
1 $\times 7$	2	$r^7 + 7r^6e + 21r^5ee = R^7 = 1,50363 = G$
2 $- r^7$	3	$7r^6e + 21r^5ee = G - r^7$
3 $\div 7r^5$	4	$re + 3ee = \frac{G - r^7}{7r^5} = D$
4 $\div$	5	$e = \frac{D}{r + 3e}$; let $r = 1$, then $D = 0,0719$

Operation $r = 1,00$
 $+ 3e = 0,18$
 Divisor $1,18)$

$0,0719 \left(\begin{array}{l} 1,00 = r \\ 0,06 = e \end{array} \right.$
 $\frac{708}{11} \frac{1,06}{11} = r + e = R$
 11 to be rejected.

Then $1 : 0,06 :: 100 : 6$ the Rate *per Cent.* required.

The first three Questions may be much more easily performed by the following Table, which is only the Amounts of one Pound for thirty-nine Years.

That

That is, of $R . RR . RRR . R^4 . R^5 .$ and so on to R^9 .

Years = t .	The Amounts of 1 <i>l.</i> at 6 per Cent. &c. Compound Interest.	Years = t .	The Amounts of 1 <i>l.</i> at 6 per Cent. &c. Compound Interest.	Years = t .	The Amounts of 1 <i>l.</i> at 6 per Cent. &c. Compound Interest.
1	1.06 = R	14	2.2609039557	27	4.8223459407
2	1.1236 = RR	15	2.3965581931	28	5.1116866971
3	1.191016 = R^3	16	2.5403516847	29	5.4183878990
4	1.26247696	17	2.6927727857	30	5.7434911729
5	1.3382255776	18	2.8543391529	31	6.0881006432
6	1.4185191122	19	3.0255995021	32	6.4533866818
7	1.5036302590	20	3.2071354722	33	6.8405898828
8	1.5938480745	21	3.3995636005	34	7.2510252757
9	1.6894789590	22	3.6035374166	35	7.6860867933
10	1.7908476965	23	3.8197496616	36	8.1472519998
11	1.8982985583	24	4.0489346413	37	8.6360871198
12	2.0121964718	25	4.2918707197	38	9.1542523470
13	2.1329282601	26	4.5493829629	39	9.7035074878

The Title of this Table shews it's Construction, and it's Use will easily appear by an Example or two.

EXAMPLE 1.

What will 375 l. 10 s. amount to in nine Years, at 6 per Cent. per Annum, &c.?

The tabular Number against 9 Years is 1,689479 which being multiplied with the Principal 375,5 will produce 634,3993 &c. viz. 634 l. 8 s. *ferè*, being the Amount or Answer required.

EXAMPLE 2.

What Principal (or Sum) must be put to Interest to raise a Stock of 634 l. 8 s. in nine Years Time, at 6 per Cent. per Annum, &c.

If the proposed Stock (viz. 634,4) be divided by the tabular Number that is against the given Number of Years (viz. 9.) the Quotient will be the Principal (or Sum) required. Viz. against 9 is 1,689479. Then 1,689479) 634,4 ($375,5 = 375$ l. 10 s. the Principal (or Sum) required.

EXAMPLE 3.

In what Time will 375 l. 10 s. raise a Stock of (or amount to) 634 l. 8 s. at 6 per Cent. &c.?

Divide

Divide the proposed Stock (*viz.* 634,4) by the given Principal (*viz.* 375,5) and the Quotient will shew the tabular Number that stands over against the Time sought. Thus

$$375,5 \overline{) 634,4} \text{ (1,689479 \&c.)}$$

This Number being sought in the Table, will be found to stand against 9 Years, which is the Time required.

But if the Quotient cannot be truly found in the Table of Amounts for Years, as above; then take out of that Table the nearest Number that is less, and make it a Divisor, by which you must divide the first Quotient; and then seek the second Quotient in the Table of Amounts for Days (which is inserted a little further on) and it will assign the Number of Days; as in this Example.

In what Time will 563 l. amount to 860 l. at 6 per Cent. per Annum, Compound Interest?

Answer. In 7 Years and 99 Days.

Thus 563) 860 (1,52753 which shews the Time to be more (or above) seven Years; for over against 7 Years is 1,50363 which being made the new Divisor: *Viz.*

$$1,50363 \overline{) 1,52753} \text{ (1,01589 \&c.)}$$

This Number is the nearest Amount to 99 Days.

Note, If the Stock, Principal, and Time be given; the Rate of Interest will be best found by extracting the Root, &c. as before in the fourth Question.

The next Thing that I shall here propose, is to make this Table (which is only calculated for the Rate of 6 per Cent.) universally useful for all the Rates of Compound Interest, *which I may presume to say, is a new Improvement of my own,* being well satisfied it never was published before; and not only so, but I have heard several very good Artists affirm it was impossible to be done.

The Method of performing it is briefly thus, Let $x =$ the Difference between $1,06 = R$, the Amount of 1 l. for one Year (in the Table), and any other proposed Amount of 1 l. for one Year; which admits of two Cases.

Case 1. If the proposed Rate be greater than the $1,06 = R$, then will $R + x =$ the true Amount of 1 l. for one Year at that Rate.

Case 2. But if the proposed Rate be less than $1,06 = R$, then it will be $R - x =$ the Amount of 1 l. &c.

$$\text{Make } \begin{cases} t - 1 = b, & t - 2 = c, & t - 3 = d, & t - 4 = f, & \&c. \\ \frac{1}{2} t b = g, & \frac{1}{4} c g = m, & \frac{1}{8} d m = n, & \frac{1}{16} f n = s, & \&c. \end{cases}$$

L 1

Then

Then will $R^t + t R^b x + g R^c x^2 + m R^d x^3 \&c.$ = the Amount of 1*l.* at the given Rate, for any Time denoted by t , in Case 1. And $R^t - t R^b x + g R^c x^2 - m R^d x^3 \&c.$ = the Amount of 1*l.* in Case 2.

Which is no more but this: Let $R + x$ or $R - x$ (which soever it is) be involved (as directed in *Sec. 5. Chap. 2.*) to the same Power or Height as the Index t the given Time in the Question denotes: rejecting all the Powers of x above xxx or $xxxx$ at most, as useless. Then multiply that Power of $R + x$ or $R - x$ into the given Principal, and their Product will be the Amount required.

An Example or two in each Case will render all easy.

E X A M P L E 1.

Suppose it were required to find what 256*l.* would amount to in fifteen Years, at 8*l.* per Cent. per Annum Compound Interest? Here $t = 15$.

First $100 : 108 :: 1 : 1,08$ the Amount of 1*l.* at 8 per Cent. Next $1,08 - 1,06 = 0,02 = x$. And $R + x = 1,08$ as in Case 1. Then $R^{15} + 15 R^{14} x + 105 R^{13} x^2 + 455 R^{12} x^3 \&c.$ = the Amount of 1*l.* for 15 Years, at 8 per Cent.

Here $x = 0,02$. $xx = 0,0004$. and $xxx = ,000008$

By the Table $R^{15} = 2,396558$

$$\text{And } \begin{cases} 15 R^{14} x &= 2,260904 \times 15 \times ,02 &= 0,678271 \\ 105 R^{13} x^2 &= 2,132928 \times 105 \times ,0004 &= 0,089583 \\ 455 R^{12} x^3 &= 2,012196 \times 455 \times ,000008 &= 0,007324 \end{cases}$$

Sum = 3,171736

Then $3,171736 \times 256 = 811,964416 = A$.

That is, 811*l.* 9*s.* 3½*d.* ferè. Which is the Answer required.

E X A M P L E 2.

What will 365*l.* amount to in seven Years at four and a half per Cent. &c.

First $100 : 104,5 :: 1 : 1,045$ the Amount of 1*l.* at 4½*l.* per Cent.

Next $1,06 - 1,045 = 0,015 = x$. Consequently $R - x = 1,045$ as in Case 2.

Then $R^7 - 7 R^6 x + 21 R^5 x^2 - 35 R^4 x^3 \&c.$ = the Amount of 1*l.* for 7 Years, at 4½ per Cent.

Here

Here $x = ,015$; $xx = ,000225$; and $xxx = 000003375$

By the Table $R^7 = + 1,503630$

$$\text{And } \begin{cases} - 7 R^6 x & = - 0,148944 \\ + 21 R^5 xx & = + 0,006323 \\ - 35 R^4 xxx & = - 0,000141 \end{cases}$$

$$R^7 - 7 R^6 x + 21 R^5 xx - 35 R^4 xxx = 1,360868$$

Then $1,360868 \times 365 = 496,71682 = A$.

That is, 496 *l.* 14 *s.* 3 $\frac{1}{4}$ *d.* is the Answer required.

If the Reason of these two Operations be but well understood, it will be very easy to conceive how to find P , the Principal, by having A , t , and x given (because R and it's Powers are always given by the Table).

For $R^t \pm t R^b x + g R^c xx \pm m R^d xxx \times P = A$ (as above).

Therefore $\frac{A}{R^t \pm t R^b x + g R^c xx \pm m R^d xxx} = P$.

Or if A , P , and t , be given, x may be found.

For $R^t \pm t R^b x + g R^c xx \pm m R^d xxx = \frac{A}{P}$. This Equa-

tion being solved (as in Chap. 10.) the Value of x will be found; and then either $R + x$, or $R - x$ will shew the Rate of Interest, &c.

But I shall leave the numerical Operations to the Learner's Practice, supposing enough done to shew how all Questions of this Kind that are limited by whole Years may be computed.

And if the Time given or sought be not terminated by whole Years, but by Weeks, Months, Quarters, or Half-Years, &c. for resolving such Questions, the best Way will be to reduce those Parts of a Year into Days; that done, find an Answer according to the Demand of the Question (and agreeing to 1 *l.* as before) for that Number of Days; and in order to that, it will be requisite to find the Amount of 1 *l.* for one Day (as in my *Compendium of Algebra*, Page 110) which I shall here insert.

Put $a =$ the Amount sought, then it will be

$$1 : a :: a : aa :: aa : aaa :: aaa : aaaa :: aaaa : aaaaa \div \text{to } a^{365}.$$

That is $\begin{cases} \text{As one Pound is to it's Amount fer one Day} :: \text{so is that} \\ \text{Amount} : \text{to the Amount of two Days} :: \text{and so is that} \\ \text{of two Days} : \text{to that of three Days. And so on in } \div \\ \text{to 365 Days.} \end{cases}$

Then the last of the Terms will be $a^{365} = 1,06$

Put	1	$r + e = a.$ And let $r = 1$
$1 @^{365}$	2	$r^{265} + 365 r^{364} e + 66430 r^{363} ee = a^{365} = 1,06$
2 in Numb.	3	$1 + 365 e + 66430 ee = 1,06$
$3 - 1$	4	$365 e + 66430 ee = 0,06$
$4 \div 66430$	5	$,00549 e + ee = 0,0000009032 = D$
$5 \div$	6	$e = \frac{D}{,00549 + e}$

Operation $,00549$
 $+ e = ,0001$

1st Divisor	$,00559$	$0,0000009032$	$(1,0000000 = r$ $0,0001598 = e$
$+ e =$	$,00015$	559	$1,0001598 = r + e = a$
2d Divisor	$,00574$	$)3442$	true to the 7th Figure
$+ e =$	$,000059$	2870	and only too much by
3d Divisor	$,005799$	$)57200$	2 in the 8th, at one
&c.		&c.	Operation.

Now $r = 1,00016$ for a second Operation. Then

2 in Numb.	7	$1,06013401407 + 386,887 e + 70402,172 ee = 1,06.$ Hence it appears that $r - e = a.$
Therefore	8	$1,06013401407 - 386,887 e + 70402,172 ee = 1,06$
$8 +$	9	$386,887 e - 70402,172 ee = 0,00013401407$
$9 \div$	10	$,0054953 - ee = ,0000000019035503$
$10 \div$	11	$e = \frac{,0000000019035503}{,0054953 - e}$

Operation $,0054953$
 $- e = 3$

1st Divisor	$,0054950$	$0,0000000019035503$	$(1,00016 = r$ $0,000000346417 = e$ $1,000159653583 = r - e$
$- e =$	34	164850	
	$,00549466$	$)2550503$	
	46	2197864	
2d Divisor	$,005494614$	$)35263900$	
$- e =$	64	32967684	
3d Divisor	$,00549460$	$)2296216$	
		2197840	
		98376	
		54946	
		&c.	

Which being further pursued to a third Operation will give
 $a = 1,000159653587453$ &c. This

This Value of a is the Amount of 1 $l.$ for one Day, from which, if 1 $l.$ be subtracted, the Remainder = ,000159653587 &c. will be the Interest of 1 $l.$ for one Day. Consequently, if any proposed Principal be multiplied into either of these, the respective Product will be the Amount or Interest of that Principal for one Day, at 6 per Cent. &c.

And that the Amount (or Interest) of any Principal or Sum may be easily computed for any Number of Days less than a Year; I have here inserted the following Table, which with a great deal of Care (and I believe Exactness) is calculated from the last found (1,000159653587453) Amount of 1 $l.$ for one Day. To which also is annexed a Table of the Amounts of 1 $l.$ for Months.

Days	Amounts of 1 $l.$ &c.	Days	Amounts of 1 $l.$ &c.	Days	Amounts of 1 $l.$ &c.
1	1.0001596536	26	1.0041592879	51	1.0081749166
2	1.0003193326	27	1.0043196055	52	1.0083358753
3	1.0004790372	28	1.0044799487	53	1.0084968597
4	1.0006387673	29	1.0046403175	54	1.0086578699
5	1.0007985229	30	1.0048007120	55	1.0088189057
6	1.0009583039	31	1.0049611320	56	1.0089799673
7	1.0011181105	32	1.0051215776	57	1.0091410545
8	1.0012779426	33	1.0052820488	58	1.0093021675
9	1.0014378002	34	1.0054425457	59	1.0094633062
10	1.0015976834	35	1.0056030682	60	1.0096244707
11	1.0017575920	36	1.0057636164	61	1.0097856608
12	1.0019175262	37	1.0059241901	62	1.0099468767
13	1.0020774859	38	1.0060847895	63	1.0101081184
14	1.0022374712	39	1.0062454146	64	1.0102693858
15	1.0023974820	40	1.0064060653	65	1.0104306789
16	1.0025575184	41	1.0065667416	66	1.0105919978
17	1.0027175803	42	1.0067274436	67	1.0107533424
18	1.0028776677	43	1.0068881712	68	1.0109147128
19	1.0030377808	44	1.0070489245	69	1.0110761090
20	1.0031979193	45	1.0072097035	70	1.0112375309
21	1.0033580850	46	1.0073705082	71	1.0113989786
22	1.0035182732	47	1.0075313385	72	1.0115604521
23	1.0036784885	48	1.0076921945	73	1.0117219513
24	1.0038387294	49	1.0078530762	74	1.0118834764
25	1.0039989958	50	1.0080139835	75	1.0120450272

Days

Days	Amounts of 1/. &c.	Days	Amounts of 1/. &c.	Days	Amounts of 1/. &c.
76	1.0122066038	116	1.0186908655	156	1.0252166658
77	1.0123682062	117	1.0188535031	157	1.0253803453
78	1.0125398344	118	1.0190161667	158	1.0255440509
79	1.0126914885	119	1.0191788563	159	1.0257077827
80	1.0128531683	120	1.0193415719	160	1.0258715406
81	1.0130148739	121	1.0195043134	161	1.0260353247
82	1.0131766054	122	1.0196670809	162	1.0261991349
83	1.0133383627	123	1.0198298745	163	1.0263629713
84	1.0135001458	124	1.0199926934	164	1.0265268338
85	1.0136619547	125	1.0201555389	165	1.0266907225
86	1.013827895	126	1.0203184110	166	1.0268546374
87	1.0139856501	127	1.0204813084	167	1.0270185784
88	1.0141475365	128	1.0206442319	168	1.0271825456
89	1.0143094488	129	1.0208071814	169	1.0273465389
90	1.0144713869	130	1.0209701569	170	1.0275105585
91	1.0146333511	131	1.0211331585	171	1.0276746046
92	1.0147953408	132	1.0212961861	172	1.0278386764
93	1.0149573565	133	1.0214592397	173	1.0280027746
94	1.0151193981	134	1.0216223193	174	1.0281668989
95	1.0152814655	135	1.0217854250	175	1.0283310494
96	1.0154435589	136	1.0219485567	176	1.0284952202
97	1.0156056781	137	1.0221117144	177	1.0286594291
98	1.0157678232	138	1.0222748982	178	1.0288236583
99	1.0159299941	139	1.0224381081	179	1.0289879137
100	1.0160921910	140	1.0226013440	180	1.0291521953
101	1.0162544138	141	1.0227646060	181	1.0293160231
102	1.0164166624	142	1.0229278940	182	1.0294908372
103	1.0165789370	143	1.0230902081	183	1.0296451975
104	1.0167412375	144	1.0232545483	184	1.0298095841
105	1.0169035638	145	1.0234179146	185	1.0299739969
106	1.0170659161	146	1.0235813069	186	1.0301384359
107	1.0172282944	147	1.0237447253	187	1.0303029012
108	1.0173906985	148	1.0239081699	188	1.0304673928
109	1.0175513086	149	1.0240716405	189	1.0306319206
110	1.0177155846	150	1.0242351372	190	1.0307964557
111	1.0178780665	151	1.0243986600	191	1.0309610251
112	1.0180405744	152	1.0245622089	192	1.0311256216
113	1.0182031083	153	1.0247257830	193	1.0312902445
114	1.0183656680	154	1.0248893851	194	1.0314548937
115	1.0185282578	155	1.0250530124	195	1.0316195692

Days

Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.
196	1.0317842709	236	1.0383939484	276	1.0450459680
197	1.0319489990	237	1.0385597318	277	1.0452128133
198	1.0321137534	238	1.0387255415	278	1.0453796853
199	1.0322785341	239	1.0388913778	279	1.0455446584
200	1.0324433410	240	1.0390572405	280	1.0457135092
201	1.0326081742	241	1.0392231298	281	1.0458804611
202	1.0327730339	242	1.0393890454	282	1.0460474397
203	1.0329379198	243	1.0395549876	283	1.0462144449
204	1.0331028321	244	1.0397209563	284	1.0463814768
205	1.0332677706	245	1.0398869515	285	1.0465484353
206	1.0334327355	246	1.0400529732	286	1.0467156206
207	1.0335977268	247	1.0402190214	287	1.0468827325
208	1.0337627444	248	1.0403850961	288	1.0470498711
209	1.0339277883	249	1.0405511973	289	1.0472170363
210	1.0340928586	250	1.0407173250	290	1.0473842283
211	1.0342579552	251	1.0408834793	291	1.0475514469
212	1.0344230782	252	1.0410496601	292	1.0477186923
213	1.0345882275	253	1.0412158674	293	1.0478859643
214	1.0347534033	254	1.0413821012	294	1.0480532631
215	1.0349186054	255	1.0415483616	295	1.0482205885
216	1.0350838338	256	1.0417146485	296	1.0483879407
217	1.0352490887	257	1.0418809620	297	1.0485553196
218	1.0354143699	258	1.0420473021	298	1.0487227252
219	1.0355796775	259	1.0422136687	299	1.0488901576
220	1.0357450115	260	1.0423800618	300	1.0490576166
221	1.0359103719	261	1.0425464815	301	1.0492251025
222	1.0360757587	262	1.0427129278	302	1.0493926150
223	1.0362411719	263	1.0428794007	303	1.0495601543
224	1.0364066116	264	1.0430459001	304	1.0497277204
225	1.0365710776	265	1.0432124261	305	1.0498953132
226	1.0367375701	266	1.0433789787	306	1.0500629327
227	1.0369030889	267	1.0435455579	307	1.0502305790
228	1.0370686342	268	1.0437121637	308	1.0503982521
229	1.0372342059	269	1.0438787961	309	1.0505659519
230	1.0373998041	270	1.0440454551	310	1.0507336786
231	1.0375654287	271	1.0442121407	311	1.0509014320
232	1.0377310793	272	1.0443788529	312	1.0510692121
233	1.0378967573	273	1.0445455918	313	1.0512370191
234	1.0380624612	274	1.0447123572	314	1.0514048529
235	1.0382241916	275	1.0448791493	315	1.0515727134

Days

Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.	Days	Amounts of 1l. &c.
316	1.0517406008	339	1.0556094165	362	1.0594924636
317	1.0519085150	340	1.0557779484	363	1.0596616154
318	1.0520764559			364	1.0598307942
319	1.0522444237	341	1.0559465071	365	1.06
320	1.0524124183	342	1.0561150927		
		343	1.0562837053		
321	1.0525804397	344	1.0564523448		
322	1.0527484880	345	1.0566210112		
323	1.0529165631				
324	1.0530846650	346	1.0567897045		
325	1.0532527937	347	1.0569584248		
		348	1.0571271720		
326	1.0534209493	349	1.0572959594		
327	1.0535891317	350	1.0574647472		
328	1.0537573410				
329	1.0539255771	351	1.0576335753		
330	1.0540938401	352	1.0578024303		
		353	1.0579713122		
331	1.0542621300	354	1.0581402211		
332	1.0544304467	355	1.0583091570		
333	1.0545987903				
334	1.0547671608	356	1.0584781199		
335	1.0549355582	357	1.0586471097		
		358	1.0588161265		
336	1.0551039824	359	1.0589851703		
337	1.0552724336	360	1.0591542411		
338	1.0554409116	361	1.0593233389		

Months

The Amounts
of 1l. at 6
per Cent.
For Months.

1

2

3

4

5

6

7

8

9

10

11

12

1.0048675505

1.0097587942

1.0146738462

1.0196128224

1.0245758394

1.0295630141

1.0345744641

1.0396103076

1.0446706634

1.0497556507

1.0548653894

1.06

The use of this Table is in all respects like that of whole Years, in finding the Amount of any given Sum for any proposed Number of Days less than a Year.

EXAMPLE I.

Suppose it were required to find the Amount of 375 l. for 210 Days, at 6 per Cent.

The Amount of 1l. for 210 Days is 1.0340928 &c. per Table. Then $1.0340928 \times 375 = 387.7848$ &c. $= 387$ l. 15 s. $8\frac{1}{4}$ d. which is the Amount required. And the rest of the Variations may be performed just as in the Examples of whole Years.

But if the Time given consists of Years, and Parts of a Year; as Quarters, Months, &c. Then reduce the odd Time or Parts of the Year into Days; and the Answer may then be found at two Operations; as in the following Example.

EXAMPLE.

Example 2. Suppose it were required to find what 265 l. would amount to in five Years and 135 Days at 6 per Cent. &c.

First, the Amount of 1 l. for $\begin{cases} 5 \text{ Years is } 1,338225, \text{ \&c.} \\ 135 \text{ Days is } 1,021785, \text{ \&c.} \end{cases}$

Then $1,338225 \times 1,021785 \times 265 \text{ l.} = 362,355232, \text{ \&c.}$ being the Amount or Answer required.

Or, if the Amount and Time are given, to find the Principal: Then multiply the Amount of 1 l. for the Years, and the Amount of 1 l. for the odd Days together: And by their Product divide the given Amount, the Quotient will be the Principal required.

Example 3. What Principal will raise a Stock of 362 l. 7 s. 1 $\frac{1}{4}$ d. Or 362,355232 l. in 5 Years and 135 Days, at 6 per Cent. &c.

The Amount of 1 for $\begin{cases} 5 \text{ Years is } 1,338225, \text{ \&c.} \\ 135 \text{ Days is } 1,021785, \text{ \&c.} \end{cases}$

Then $1,338225 \times 1,021785 = 1,367378, \text{ \&c.}$ the Divisor. Next $1,367378 \mid 362,355232 = A$ (265 l. the Principal required.

Again, if the Principal and its Amount are given, to find the Time, at 6 per Cent. &c. you must divide the Amount by its Principal, and then proceed as in the Third Example, Page 256, for the Answer required.

But if the Amount and its Principal, with the Time of its being at Interest, are given, to find the Rate of Interest: Then proceed as in the Fourth Question, Page 255, &c.

Now in order to make this Table of Amounts for Days useful for all Rates of Interest (as before in that for Years) you must first find the Simple Interest of 1 l. for one Day, both at the given Rate, and also at 6 per Cent. And call their Difference x .

Thus, suppose the given Ratio were 8 per Cent. per Annum, First $130 : 8 :: 1 : 0,08$ And $100 : 6 :: 1 : 0,06$ the Two Simple Interests for one Year.

Then $365 \mid 0,08$ (0,00021917, &c. the Simple Interest of 1 l. for one Day, at 8 per Cent.

And $365 \mid 0,06$ (0,00016438, &c. the Simple Interest of 1 l. for one Day, at 6 per Cent.

Their Difference $0,00005479 = x$, which may do indifferently well for ordinary small Questions: But where Exactness is required, it will be convenient to make Use of this Proportion:

M m

Viz.

Viz. $\left\{ \begin{array}{l} \text{As the Simple Interest of } 1 \text{ l. for one Day at } 6 \text{ per Cent. :} \\ \text{Is to the Tabular Interest of } 1 \text{ l. for one Day : : So is the} \\ \text{Simple Interest of } 1 \text{ l. for one Day, at any given Rate :} \\ \text{To a Fourth Number.} \end{array} \right.$

That is, $0,00016438 : 0,00015965 :: 0,00021917 : 0,00021286$
 Then $0,00021286 - 0,00015965 = 0,00005321 = x$.

This x being *involved* with the respective *Amounts* for Days, in the same Manner as was done with those for Years (*vide* Page 258) the Result will be the *Answer* to the *Question*.

Sect. 2. *Annuities or Pensions in Arrear, computed at Compound Interest.*

When *Annuities*, &c. are said to be in *Arrear*, see Page 248. And I shall here make Use of the same Letters to represent the same Things as before in that Page, save only that R is here equal to the *Amount* of 1 l. as in *Section* 1. of this Chapter.

Suppose $u =$ the First Year's Rent of any Annuity without Interest.

Then will $Ru + u = \left\{ \begin{array}{l} \text{the Amount of the First Year's Rent, and} \\ \text{its Interests; More the 2d Year's Rent.} \end{array} \right.$

And $RRu + Ru + u = \left\{ \begin{array}{l} \text{the Amount of the 1st and 2d Years} \\ \text{Rents, with their Interests; More the} \\ \text{3d Year's Rent, \&c.} \end{array} \right.$

Here $RRu + Ru + u = A$, the *Amount* of any Yearly Rent or Annuity, being forborne Three Years. And from hence may be deduced these Proportions :

Viz. $u : Ru :: Ru : RRu :: RRu : RRRu$, and so on in $\div$ for any Number of Terms or Years denoted by t , wherein the last Term will always be uR^{t-1} .

Consequently, $A - uR^{t-1} =$ the Sum of all the Antecedents And $A - u =$ the Sum of all the Consequents in the Series.

And therefore it would be $u : uR :: A - uR^{t-1} : A - u$, Vide Page 188.

Ergo $Au - uu = RuA - uuR^t$, which, being divided all by u , will become $A - u = RA - uR^t$.

From this last Equation it will be easy to raise the following Theorems :

Theorem 1. $\left\{ \frac{uR^t - u}{R - 1} = A \right.$ Theorem 2. $\left\{ \frac{RA - A}{R - 1} = u \right.$ Theorem

Theorem 3. $\left\{ \frac{RA + u - A}{u} = R^t \right.$ If this Equation be continually divided by R , until nothing remain, the Number of those Divisions will be t . See Page 255.

Theorem 4. $\left\{ \frac{A}{u} R - R^t = \frac{A - u}{u} \right.$ If this Equation be resolved into Numbers, according to the Method proposed in Sect. 3. Chap. 10. the Root will shew the Value of R .

QUESTION 1. If 30 l. Yearly Rent, or Annuity, &c. be forborne (i. e. remain unpaid) Nine Years; what will it amount to, at 6 per Cent. per Annum, Compound Interest?

Here is given $u = 30$, $t = 9$, and $R = 1,06$; to find A . per Theorem 1.

$$R^9 = 1,689479 \text{ By the Table of Amounts for Years } 30 = u$$

$$\begin{array}{r} R^9 u = 50,684370 \\ - u = 30, \\ \hline \end{array}$$

$R - 1 = 0,06$ $20,684370$ ($344,7395 = 344 \text{ l. } 14 \text{ s. } 9 \frac{1}{2} \text{ d.} = A$, the Amount required.

QUESTION 2. What Yearly Rent or Annuity, &c. being forborne or unpaid Nine Years, will raise a Stock of 344 l. 14 s. 9 $\frac{1}{2}$ d. = 344,7395, at 6 per Cent. &c.

Here is given $A = 344,7395$, $t = 9$, and $R = 1,06$; to find u . per Theorem 2.

$$\begin{array}{r} AR = 344,7395 \times 1,06 = 365,42387 \\ - A = 344,7395 \\ \hline \end{array}$$

$$R^t - 1 = 1,689479 - 1 = 0,689479 \quad 20,68437 \quad (30 = u.$$

QUESTION 3. In what Time will 30 l. Yearly Rent raise a Stock or Amount to 344 l. 14 s. 9 $\frac{1}{2}$ d. allowing 6 per Cent. for the Forbearance of Payments?

Here is given $u = 30$, $A = 344,7395$, and $R = 1,06$; to find t . per Theorem 3.

First $AR + u - A = 365,42387 + 30 - 344,7395 = 50,68437$. And $u = 30$ $50,68437$ ($1,689479 = R^t$. Then $R = 1,06$ $1,689479$ ($1,593848$. And $1,06$ $1,593848$ ($1,50363$; and so on until it become $1,06$ $1,06$ (1 . which will be at the Ninth Division; therefore $t = 9$.

M m 2

Or

Or $R = 1,689,479$, being sought in the *Table of Amounts* for *Years*, will be found to stand over-against 9 *Years*, which is the *Time required*.

QUESTION 4. If 30 *l.* per Annum, being unpaid Nine Years, will amount to 344 *l.* 14 *s.* 9 *d.* allowing Compound Interest for every Payment as it becomes due, What must the Rate of Interest be per Cent. &c.

Here is given $u = 30$, $A = 344,7395$, and $t = 9$; to find R by the last of the Four *Æquations*, Viz. $\left\{ \frac{A}{u} R - R^t = \frac{A - u}{u} \right.$

First $\frac{A}{u} = \frac{344,7395}{30} = 11,491317$. And $\frac{A - u}{u} = 10,491317$.

Hence there is this *Æquation*; $11,491317 R - R^9 = 10,491317$.

Let	1	$r + e = R$, and suppose $r = 1$
1 $\odot$ 9	2	$r^9 + 9r^8e + 36r^7ee = R^9$
1 $\times \frac{2}{u}$ in Numb.	3	$11,491317 + 11,491317e = 11,491317 R$
2 in Numb.	4	$1,000000 + 9,000000e + 36ee = R^9$
3 $-$ 4	5	$10,491317 + 2,491317e - 36ee = 10,491317$
Whence	6	$36ee = 2,491317e$
6 $\div$ 36 e	7	$e = 0,06$, &c.

First $r = 1$
 $+ e = 0,06$ } $= 1,06 = R$ { *As may be easily try'd by involving it, and ordering it, as the Æquation above directs.*

Section 3. To find the **Present Worth** of Annuities, Pensions, or Leases, &c. at Compound Interest.

Let $P =$ the *present Worth* of any Annuity, or Lease, &c. and the rest of the Letters as before.

Then, from what has been said in *Section 3. Chap. II.* about *Purchasing of Annuities*, &c. at Simple Interest, it will be easy to form the like *Theorems* here at Compound Interest, viz. by Combining *Theorem 1. Page 266.* and *Theorem 1. Page 254.* into one *Theorem*.

For $\left\{ \frac{uR^t - u}{R - 1} = A \right\}$ *The Amount of any Yearly Rent being unpaid any Number of Years. Per Theorem 1. of the last Section. Page 266.*

And $PR^t = A$ { *The Amount of any Principal or Sum being put to Interest, for the same Number of Years. Per Theorem 1. Page 254.*

Hence

Hence it follows, That $P R^t = \frac{u R^t - u}{R - 1}$,

Viz. $P R^{t+1} - P R^t = u R^t - u$ being the very same Equation with that in my *Compendium of Algebra*, Page 112. which is there raised from the Consideration of purchasing *Annuities*, or taking of *Leases*, &c. to be grounded upon a *Rank* or *Series* of *Geometrical Proportionals* continually decreasing. Thus $\frac{u}{R}$ is the *First* and *Greatest Term*; R the common *Ratio* of all the *Terms*; and P is the *Sum* of all the *Series*.

That is, $\frac{u}{R} : \frac{u}{RR} :: \frac{u}{RR} : \frac{u}{RRR} :: \frac{u}{RRR} : \frac{u}{R^4} :: \frac{u}{R^4} : \frac{u}{R^5}$, &c. in $\div$ until the last *Term* $= \frac{u}{R^t}$. Then will $P - \frac{u}{R^t}$ be the *Sum* of all the *Antecedents*, and $P - \frac{u}{R}$ the *Sum* of all the *Consequents*.

Therefore it will be

$$\frac{u}{R} : \frac{u}{RR} \text{ Or (in the same Ratio) } u : \frac{u}{R} :: P - \frac{u}{R^t} : P - \frac{u}{R},$$

which produces $P R^{t+1} - u R^t = P R^t - u$. As above.

From this Equation may be deduced the following Theorems:

$$\text{Theorem 1. } \left\{ \frac{u - \frac{u}{R^t}}{R - 1} = P. \text{ Theorem 2. } \left\{ \frac{P R^t \times R - P R^t}{R - 1} = u. \right. \right.$$

$$\text{Theorem 3. } \left\{ \frac{u}{P + u - P R} = R^t \right\} \text{ Which, being continually divided by } R, \text{ will give } t.$$

$$\text{Theorem 4. } \left\{ \frac{u}{P} = \frac{u}{P} R^t + R^t - R^{t+1}. \right. \text{ The Resolving of which Equation will discover the Value of } R.$$

Question 1. What is 30 l. Yearly Rent, to continue Seven Years, worth in ready Money, allowing 6 per Cent. Compound Interest to the Purchaser?

Here is given $u = 30$. $t = 7$. And $R = 1.06$ to find P . per Theorem 1. Viz. $\frac{u}{R^t} = \frac{30}{1.50363} = 19.9517$.

$$\text{And } 30 - 19.9517 = 10.0483 = u - \frac{u}{R^t}.$$

Then

Then $R - 1 = 0,06$ $10,0483$ ($167,4716 = P = 167 \text{ l. } 9 \text{ s. } 5 \text{ d.}$ being the Answer required.

Question 2. *What Annuity or Yearly Rent, to continue Seven Years, may be purchased for 167 l. 9 s. 5 d. allowing 6 per Cent. Compound Interest to the Purchaser?*

In this Question there is given $P = 167,4716$. $t = 7$
And $R = 1,06$ to find u . By the Second Theorem.

$$\text{First } P R^t \times R = 251,8153 \times 1,06 = 266,9242$$

$$\text{And } - P R^t = 167,4716 \times 1,50363 = 251,8153$$

$$\text{Then } R^t - 1 = 0,50363 \quad 15,1089 \quad (30 = u)$$

That is $u = 30 \text{ l.}$ the Answer required.

Question 3. *How long may one have a Lease of 30 l. Yearly Rent, for 167 l. 9 s. 5 d. allowing 6 per Cent. Compound Interest to the Purchaser?*

Here is given $P = 167,4716$. $u = 30$. And $R = 1,06$ to find t . By the Third Theorem.

$$\text{First } P + u = 167,4716 + 30 = 197,4716$$

$$\text{And } - P R = 177,5199$$

$$\text{Then } 19,9517) 30 = u (1,50363 = R^t$$

If this $1,50363 = R^t$ be either continually divided by $1,06 = R$ until nothing remain (As before in Page 255.) Or if it be sought in the Table of Amounts for Years, &c. it will discover $t = 7$ which is the true Answer required.

Question 4. *Suppose one should give 167 l. 9 s. 5 d. for the Purchase of a Pension, or Annuity of 30 l. per Annum, to continue Seven Years: At what Rate of Interest, per Cent. would that Purchase be made, allowing Compound Interest to the Purchaser?*

In this Question there is given, $P = 167,4716$. $u = 30$ and $t = 7$ to find R . Per Theorem 4 in this Equation $\left\{ \frac{u}{P} = \frac{u}{P} \right.$

$R^t + R^t - R^{t+1}$, which being brought into Numbers, and its Root extracted, as in the fourth Question of the last Section; the Value of R will be found $1,06$, and then it will be $1 : 0,06 :: 100 : 6$, the Rate per Cent. as was required.

These

These Four *Questions* include all the *Varieties* that can be proposed about purchasing *Annuities* or *Leases*, &c. which are to be either immediately enter'd upon, or in Possession at the Time when the *Purchase* is made.

But such *Questions* as relate to *Annuities*, or a taking of *Leases*, &c. in *Reversion*, must be parted or divided into two distinct *Questions*, each to be separately consider'd by itself (See Page 252.) As in the following *Examples* :

Example 1. Suppose it were required to compute the present Worth of 75 l. Yearly Rent, which is not to commence or be enter'd upon, until Ten Years hence; and then to continue Seven Years after that Time: at 6 per Cent. &c. Compound Interest?

The First Work in this *Question* is, to find what 75 l. per Annum, to continue Seven Years, is worth in ready Money; as if it were to be immediately enter'd upon: And to perform that, there is given $u = 75$. $R = 1,06$. and $t = 7$. to find P . as in the First *Question* of this *Section*.

$$\text{Thus, } \frac{u}{R^t} = \frac{75}{1,50363} = 49,8793 \text{ And } 75 - 49,8793 = 25,1207 \\ = u - \frac{u}{R^t}$$

Then, $R - 1 = 0,06$ $25,1207 = 418,6783 = 418 \text{ l. } 14 \text{ s. } 6 \frac{3}{4} \text{ d.}$ the Answer to the First Part of the *Question*.

Then the next Work will be, to find what *Principal* or *Sum* being put out Ten Years, at 6 per Cent. &c. will amount to 418 l. 14 s. 6 $\frac{3}{4}$ d. Here is given $A = 418,6783$, $R = 1,06$, $t = 10$. to find P . Per *Theorem* 2. Page 254.

Thus $R^{10} = 1,790847$ $418,6783 = A$ $(233,7884 = 233 \text{ l. } 15 \text{ s. } 9 \text{ d.})$ the present Worth of 75 l. per Annum in *Reversion*, &c. As was required.

Example 2. What *Annuity* or *Yearly Rent*, to be enter'd upon Ten Years hence, and then to continue Seven Years, may be purchased for 233 l. 15 s. 9 d. Ready Money, at 6 per Cent. &c. Compound Interest?

In the 1st Work of this *Question* there is given, $P = 233,7884$ $R = 1,06$. And $t = 10$ (the Time which the *Annuity* is not to be enter'd upon) to find A . Per *Theorem* 1. Page 254.

$$\text{Thus, } P R^t = 233,7884 \times 1,790847 = 418,6783 = A \text{ the Amount}$$

Amount of 233*l.* 15*s.* 9*d.* put to Interest Ten Years, at 6 per Cent. &c. Then, for the Second Work of the *Question*, there is given $P = 418,6783$. $R = 1,06$. And $t = 7$ (the Time that the Annuity is to be enjoy'd) to find u . Per Theorem 2. of this Section.

$$\text{Thus } P R^t \times R = 418,6783 \times 1,50363 \times 1,06 = 667,3095$$

$$- P R^t = 418,6783 \times 1,50363 = 629,5372$$

$$R^t - 1 = 0,50363 \quad 37,7743(75 = u$$

That is, $u = 75$ *l.* the Yearly Rent required by the *Question*.

These Two Examples of finding P and u do fully shew the Method that must be used in Resolving the two General, and indeed, the most useful *Questions* about Annuities or Leases in Reversion: And if there be Occasion, either the Rate, or the Time, viz. R or t , may be found by a due Application of their respective Theorems.

Note, That which hath been done in the two last Sections about Annuities or Yearly Rents, &c. at 6 per Cent. may also be done for any Rate of Interest, by applying the Difference of the Rates (viz. x .) As directed in the First Section of this Chapter.

Now because that Rents and Annuities, &c. are usually paid either by Quarterly or Half Yearly Payments, and the Method of computing them by the Pen may be thought a little troublesome; I have inserted the following Tables of the Amounts of 1*l.* for each, at 6 per Cent.

Half Years = <i>l.</i>	Annuities of 1 <i>l.</i> at 6 per Cent. Com- pound Interest.	Half Years = <i>l.</i>	Annuities of 1 <i>l.</i> at 6 per Cent. Com- pound Interest.	Half Years = <i>l.</i>	Annuities of 1 <i>l.</i> at 6 per Cent. Com- pound Interest.
1	1,0295630141	11	1,3777875592	21	1,8437905523
2	1,06	12	1,4185191122	22	1,8982985583
3	1,0613367949	13	1,4604548127	23	1,9544179853
4	1,1236	14	1,5036302590	24	2,0121964718
5	1,1568170026	15	1,5480821017	25	2,0716830644
6	1,191016	16	1,5938480745	26	2,1329282601
7	1,2262260228	17	1,6409670276	27	2,1959840483
8	1,26247696	18	1,6894789589	28	2,2609039557
9	1,2997995842	19	1,7394250493	29	2,3277430912
10	1,3382255776	20	1,7908476965	30	2,3965581001

Quarterly Amounts.

Quarters of a Year = $\frac{1}{4}$.	Amounts of 1 <i>l</i> . at 6 per Cent. &c. Compound Interest.	Quarters of a Year = $\frac{1}{4}$.	Amounts of 1 <i>l</i> . at 6 per Cent. &c. Compound Interest.	Quarters of a Year = $\frac{1}{4}$.	Amounts of 1 <i>l</i> . at 6 per Cent. &c. Compound Interest.
1	1,0146738461	21	1,3578024938	41	1,8171263199
2	1,0295630141	22	1,3777875592	42	1,8437905523
3	1,0446706634	23	1,3980050019	43	1,8708460509
4	1,06	24	1,4185191122	44	1,8982985583
5	1,0755542769	25	1,4393342435	45	1,9261538989
6	1,0913367949	26	1,4604548127	46	1,9544179853
7	1,1073509032	27	1,4818853020	47	1,9830968140
8	1,1236	28	1,5036302590	48	2,0121964718
9	1,1400875335	29	1,5256942978	49	2,0417231330
10	1,1568170026	30	1,5480821017	50	2,0716830644
11	1,1737919574	31	1,5707984203	51	2,1020826228
12	1,191016	32	1,5938480745	52	2,1329282601
13	1,2084927856	33	1,6172359557	53	2,1642265211
14	1,2262260228	34	1,6409670276	54	2,1959840483
15	1,2442194748	35	1,6650463253	55	2,2282075801
16	1,26247696	36	1,6894789589	56	2,2609039557
17	1,2810023527	37	1,7142701133	57	2,2940801123
18	1,2997995842	38	1,7394250493	58	2,3277430912
19	1,3188726433	39	1,7649491048	59	2,3619000349
20	1,3382255776	40	1,7908476965	60	2,3965581931

Either of these *Tables* may also be made useful for any proposed *Rate of Interest*; by making the $\frac{1}{2}$ or $\frac{1}{4}$ of the *Difference* of the *Rate* = x , &c.

As for Instance, suppose any of the aforesaid *Questions* about *Annuities* or *Rents*, &c. were to be computed at 8 per Cent. per Ann.

Then $1,08 - 1,06 = 0,02 = x$ for *Yearly Payments*; as before. Consequently 2) $0,02$ ($0,01 = x$ for *Half Year's Payments*).

Or 4) $0,02$ ($0,005 = x$ for *Quarterly Payments*).

Now these *Values* of x , although they are not really true, yet they may serve indifferently well for small *Rents*; as I have already said, *Page* 265. But if you would work exactly;

Then $\sqrt{1,08} = 1,0392304845$, &c.

— $\sqrt{1,06} = 1,0295680141$, *Vide Table*, *Page* 272.

Difference = $0,0096624704 = x$ for *Half Yearly Payments*.

N n

And

And $\sqrt{} : \sqrt{} 1,08 = 1,0194263092, \&c.$

— $\sqrt{} : \sqrt{} 1,06 = 1,0146738461.$ See the *Last Table*.

Their Difference $0,0047524631 = x$, for *Quarterly Payments*.

These are the true *Values* of x , which being *involved* with their respective *Amounts* (as before for *Years*, &c.) according as the *Question* requires, the *Result* will be the *Answer* at 8 *per Cent*. &c. The like may be done for any other *Rate*, either *Greater* or *Less* than 6.

Now, although the Method used here (and in *Page 257* and *258*, &c.) be really true (by which the *Tables* calculated only for 6 *per Cent*. are made effectual for all *Rates* of *Compound Interest*) yet it was rather propos'd to shew what may possibly be performed by the Pen, without a great many *Tables* of several *Rates*, than intended for common Practice.

For it must needs be confess'd, that *Tables*, calculated on Purpose for any designed *Rate* of *Interest*, are much more ready and useful in common Practice. And therefore since the Legislative Power hath thought fit to reduce the *Rate* of *Interest*, and hath settled it by an Act of Parliament, at 5 *per Cent*. I've therefore been at the Trouble (*which was not a little*) to calculate the following *Tables* for that *Rate*; but don't think it convenient to take the *Tables* at 6 *per Cent*. out of the Book, because the Examples are all suited to them; and not only so, but they may be found useful in the taking of *Leases* for Houses, &c. For in those Cases, the *Purchaser* is allowed more *Interest* for his purchase Money, than the common *Rate* paid upon the Loan of Money.

Here

Here follow New Tables of the *Amounts* of one Pound at the Rate of 5 per Cent. per Annum Compound Interest. For Years, Half Years, Quarters, Months, and Days.

I. The Table of the Yearly Amounts of 1 l. &c.

Years l.	The Amounts of 1 l. &c.	Years l.	The Amounts of 1 l. &c.	Years l.	The Amounts of 1 l. &c.
1	1,05 = R	14	1,97993160	27	3,73345632
2	1,1025 = R R	15	2,07892818	28	3,92012914
3	1,157625 = R ³	16	2,18287459	29	4,11613599
4	1,21550625	17	2,29201832	30	4,32194239
5	1,27628156	18	2,40661923	31	4,53803949
6	1,34009564	19	2,52695019	32	4,76494147
7	1,40710042	20	2,65329770	33	5,00318854
8	1,47745544	21	2,78596259	34	5,25334797
9	1,55132822	22	2,92526072	35	5,51601536
10	1,62889463	23	3,07152375	36	5,79181613
11	1,71033936	24	3,22509994	37	6,08140694
12	1,79585633	25	3,38635494	38	6,38547729
13	1,88564914	26	3,55567269	39	6,70475115

II. The Table of the Half Yearly Amounts of 1 l. &c.

Half Yrs. l.	The Amounts of 1 l. &c.	Half Yrs. l.	The Amounts of 1 l. &c.	Half Yrs. l.	The Amounts of 1 l. &c.
1	1,02469507	11	1,30779943	21	1,66912030
2	1,05	12	1,34009564	22	1,71033936
3	1,07592983	13	1,37318940	23	1,75257632
4	1,1025	14	1,40710042	24	1,79585633
5	1,12972632	15	1,44184887	25	1,84020513
6	1,157625	16	1,47745544	26	1,88564914
7	1,18621264	17	1,51394132	27	1,93221539
8	1,21550625	18	1,55132822	28	1,97993160
9	1,24552327	19	1,58963838	29	2,02882616
10	1,27628156	20	1,62889463	30	2,07892818

III. The *Table of the Quarterly Amounts of 1 l. &c.*

Quarters = l.	The Amounts of 1 l. &c.	Quarters = l.	The Amounts of 1 l. &c.	Quarters = l.	The Amounts of 1 l. &c.
1	1,01227223	21	1,29194439	41	1,64888480
2	1,02169507	22	1,30779943	42	1,66912031
3	1,03727037	23	1,32384905	43	1,68960414
4	1,05	24	1,34009564	44	1,71033936
5	1,06288585	25	1,35654161	45	1,73132904
6	1,07592983	26	1,37318940	46	1,75257632
7	1,08913389	27	1,39004151	47	1,77408435
8	1,1025	28	1,40710042	48	1,79585633
9	1,11603014	29	1,42436869	49	1,81789549
10	1,12972632	30	1,44184887	50	1,84020513
11	1,14359059	31	1,45954358	51	1,86278856
12	1,157625	32	1,47745544	52	1,88564914
13	1,17183164	33	1,49558712	53	1,90879027
14	1,18621264	34	1,51394132	54	1,93221539
15	1,20077012	35	1,53252076	55	1,95592799
16	1,21550625	36	1,55132822	56	1,97993160
17	1,23042323	37	1,57036648	57	2,00422978
18	1,24552327	38	1,58963838	58	2,02882616
19	1,26080862	39	1,60914680	59	2,05372439
20	1,27628156	40	1,62889463	60	2,07892818

IV. The *Table of the Monthly Amounts of 1 l. &c.*

Months = l.	The Amounts of 1 l. &c.	Months = l.	The Amounts of 1 l. &c.	Months = l.	The Amounts of 1 l. &c.
1	1,00407412	5	1,02053728	9	1,03727037
2	1,00816485	6	1,02469507	10	1,04149634
3	1,01227223	7	1,02886981	11	1,04573953
4	1,01639636	8	1,03306155	12	1,05

NOTE: The *Amount of one Pound, for one Day, is 1,0001336807225, &c.* (found as that in *Page 260*) but in the following *Table*, I take only Nine of those *Figures*, as being sufficient in Practice, for computing the *Interest* of any *Sum* not exceeding One Hundred Millions of Pounds.

V. The

V. The Table of the Daily Amounts of 1 l. &c.

Days= l.	The Amounts of 1 l. &c.	Days= l.	The Amounts of 1 l. &c.	Days= l.	The Amounts of 1 l. &c.
1	1,00013368	36	1,00482376	71	1,00953587
2	1,00026738	37	1,00495810	72	1,00967082
3	1,00040109	38	1,00509245	73	1,00980579
4	1,00053483	39	1,00522681	74	1,00994079
5	1,00066858	40	1,00536119	75	1,01007579
6	1,00080235	41	1,00549558	76	1,01021083
7	1,00093614	42	1,00563000	77	1,01034587
8	1,00106994	43	1,00576443	78	1,01048093
9	1,00120377	44	1,00589888	79	1,01061602
10	1,00133761	45	1,00603335	80	1,01075112
11	1,00147147	46	1,00616784	81	1,01088623
12	1,00160535	47	1,00630234	82	1,01102137
13	1,00173924	48	1,00643687	83	1,01115652
14	1,00187315	49	1,00657141	84	1,01129169
15	1,00200708	50	1,00670597	85	1,01142688
16	1,00214103	51	1,00684055	86	1,01156209
17	1,00227500	52	1,00697514	87	1,01169732
18	1,00240899	53	1,00710975	88	1,01183256
19	1,00254299	54	1,00724438	89	1,01196783
20	1,00267701	55	1,00737903	90	1,01210311
21	1,00281105	56	1,00751370	91	1,01223841
22	1,00294510	57	1,00764839	92	1,01237372
23	1,00307918	58	1,00778309	93	1,01250906
24	1,00321327	59	1,00791781	94	1,01264441
25	1,00334738	60	1,00805255	95	1,01277978
26	1,00348151	61	1,00818731	96	1,01291517
27	1,00361565	62	1,00832208	97	1,01305058
28	1,00374982	63	1,00845687	98	1,01318600
29	1,00388400	64	1,00859168	99	1,01332145
30	1,00401820	65	1,00872651	100	1,01345691
31	1,00415242	66	1,00886136	101	1,01359239
32	1,00428665	67	1,00899623	102	1,01372788
33	1,00442091	68	1,00913111	103	1,01386340
34	1,00455518	69	1,00926601	104	1,01399893
35	1,00468947	70	1,00940093	105	1,01413448

Days

Days= i.	The Amounts of 1 l. &c.	Days= i.	The Amounts of 1 l. &c.	Days= i.	The Amounts of 1 l. &c.
106	1,01427005	146	1,01970775	186	1,02517459
107	1,01440564	147	1,01984406	187	1,02531164
108	1,01454125	148	1,01998039	188	1,02544870
109	1,01467687	149	1,02011675	189	1,02558578
110	1,01481252	150	1,02025312	190	1,02572288
111	1,01494818	151	1,02038950	191	1,02586000
112	1,01508386	152	1,02052591	192	1,02599714
113	1,01521955	153	1,02066234	193	1,02613430
114	1,01535527	154	1,02079878	194	1,02627147
115	1,01549100	155	1,02093524	195	1,02640866
116	1,01562675	156	1,02107172	196	1,02654588
117	1,01576252	157	1,02120822	197	1,02668310
118	1,01589831	158	1,02134473	198	1,02682015
119	1,01603412	159	1,02148127	199	1,02695762
120	1,01616994	160	1,02161782	200	1,02709490
121	1,01630578	161	1,02175439	201	1,02723221
122	1,01644164	162	1,02189098	202	1,02736953
123	1,01657752	163	1,02202758	203	1,02750686
124	1,01671349	164	1,02216421	204	1,02764422
125	1,01684933	165	1,02230085	205	1,02778160
126	1,01698527	166	1,02243751	206	1,02791899
127	1,01712122	167	1,02257419	207	1,02805640
128	1,01725719	168	1,02271089	208	1,02819384
129	1,01739317	169	1,02284761	209	1,02833129
130	1,01752918	170	1,02298434	210	1,02846875
131	1,01766521	171	1,02312109	211	1,02860624
132	1,01780125	172	1,02325787	212	1,02874375
133	1,01793731	173	1,02339466	213	1,02888127
134	1,01807338	174	1,02353147	214	1,02901881
135	1,01820948	175	1,02366829	215	1,02915637
136	1,01834559	176	1,02380514	216	1,02929395
137	1,01848173	177	1,02394200	217	1,02943154
138	1,01861788	178	1,02407888	218	1,02956916
139	1,01875405	179	1,02421578	219	1,02970679
140	1,01889024	180	1,02435270	220	1,02984445
141	1,01902644	181	1,02448964	221	1,02998212
142	1,01916267	182	1,02462659	222	1,03011980
143	1,01929891	183	1,02476356	223	1,03025751
144	1,01943517	184	1,02490055	224	1,03039524
145	1,01957145	185	1,02503756	225	1,03053298

Days

Days= i.	The Amounts of 1 l. &c.	Days= i.	The Amounts of 1 l. &c.	Days= i.	The Amounts of 1 l. &c.
226	1,03067074	266	1,03619636	306	1,04175160
227	1,03080852	267	1,03633488	307	1,04189086
228	1,03094632	268	1,03647342	308	1,04203015
229	1,03108414	269	1,03661197	309	1,04216944
230	1,03122197	270	1,03675055	310	1,04230876
231	1,03135983	271	1,03688914	311	1,04244810
232	1,03149770	272	1,03702775	312	1,04258745
233	1,03163559	273	1,03716638	313	1,04272683
234	1,03177350	274	1,03730503	314	1,04286622
235	1,03191143	275	1,03744370	315	1,04300563
236	1,03204938	276	1,03758239	316	1,04314506
237	1,03218734	277	1,03772109	317	1,04328451
238	1,03232533	278	1,03785982	318	1,04342397
239	1,03246333	279	1,03799856	319	1,04356346
240	1,03260135	280	1,03813732	320	1,04370297
241	1,03273939	281	1,03827609	321	1,04384249
242	1,03287744	282	1,03841489	322	1,04398203
243	1,03301552	283	1,03855371	323	1,04412159
244	1,03315361	284	1,03869254	324	1,04426117
245	1,03329173	285	1,03883139	325	1,04440077
246	1,03342986	286	1,03897027	326	1,04454038
247	1,03356801	287	1,03910916	327	1,04468002
248	1,03370617	288	1,03924817	328	1,04481967
249	1,03384436	289	1,03938699	329	1,04495934
250	1,03398157	290	1,03952594	330	1,04509903
251	1,03412079	291	1,03966491	331	1,04523874
252	1,03425903	292	1,03980389	332	1,04537847
253	1,03439729	293	1,03994289	333	1,04551822
254	1,03453557	294	1,04008191	334	1,04565798
255	1,03467387	295	1,04022095	335	1,04579777
256	1,03481218	296	1,04036001	336	1,04593757
257	1,03495052	297	1,04049908	337	1,04607739
258	1,03508887	298	1,04063818	338	1,04621723
259	1,03522724	299	1,04077729	339	1,04635709
260	1,03536563	300	1,04091642	340	1,04649697
261	1,03550404	301	1,04105557	341	1,04663686
262	1,03564247	302	1,04119474	342	1,04677678
263	1,03578091	303	1,04133393	343	1,04691671
264	1,03591938	304	1,04147314	344	1,04705667
265	1,03605786	305	1,04161236	345	1,04719664

Days

Days= 1.	The Amounts of 1 l. &c.	Days= 1.	The Amounts of 1 l. &c.	Days= 1.	The Amounts of 1 l. &c.
346	1,04733663	353	1,04831708	360	1,04929845
347	1,04747664	354	1,04845722	361	1,04943872
348	1,04761666	355	1,04859738	362	1,04957901
349	1,04775671	356	1,04873756	363	1,04971932
350	1,04789677	357	1,04887775	364	1,04985965
351	1,04803686	358	1,04901797	365	1,04999999
352	1,04817696	359	1,04915820	366	1,05

I think it is needless to say any Thing of the Use of these *Tables*, because I take it for granted, that whoever understands the Work of the foregoing *Examples*, at 6 per Cent. cannot but know how to make Use of these *Tables* at 5 per Cent. as Occasion requires.

Thus far concerning *Annuities*, or *Leases*, &c. that are *limited* by any assigned *Time*; and 'tis only such that can be computed by *Theorems* or certain *Rules*. However, it may not perhaps be *unacceptable*, to insert a brief Account of some *Estimates* that have been reasonably made, by *two* very ingenious *Persons*, about the Proportion or Difference of *Mens Lives*, according to their several *Ages*; which may be of good Use in computing the *Values* of *Annuities*, or taking of *Leases* for *Lives*, &c.

Sir William Petty, in his Discourse made before the *Royal Society* (*Anno* 1674) concerning the Use of DUPLICATE PROPORTION, in the Life of *Man* and its Duration, saith, that it's found by *Experience* there are more Persons living of between 16 and 26 *Years Old*, than of any other *Age* or *Decade* of *Years* in the whole Life of *Man* (*viz.* 70 or 80 *Years.*) His Reason for that Assertion I shall omit; but supposing it true, he thence infers, that the *Roots* of every Number of *Mens Ages* under 16 (whose *Root* is 4) compared with the said Number 4, doth shew the *Proportion* of the Likelihood of such *Mens* reaching the *Age* of 70 *Years*.

As for *Example*, 'tis 4 *Times* more likely, that one of 16 *Years Old* should live to 70, than a New-born *Babe*: 'Tis 3 *Times* more likely, that one of 9 *Years Old* should attain the *Age* of 70, than the said *Infant*, &c.

On the other Hand, 'tis 5 to 4, that one of 25 *Years Old* will die before one of 16: And 6 to 5, that one of 36 will die before one of 25. And so on according to the *Roots* of any other declining *Age*, compared with the (4,6) the *Root* of 21, which is the *Year* of *Perfection* according to the Sense of our *Law*, and the *Age* for whose Life a *Lease* is most *valuable*.

2. *The ingenious and great Mathematician, Doctor Edmund Halley (in Philosoph. Transact. Numb. 196) doth, with great Industry and Skill, draw an Estimate of the Proportion of Mens Lives, from the Monthly Tables of the Births and Funerals in Breslaw, the Capital City of the Province of Silesia; or, as the Germans call it, Schlesia. Whence he proves that it's 80 to 1, a Person of 25 Years Old will not die in a Year: That it is $5\frac{1}{2}$ to 1, that a Man of 40 will live 7 Years: That a Man of 30 Years Old may reasonably expect to live 27 or 28 Years, &c.*

Now from these and the like *Proportions* (he justly infers) that the *Price of Insurance* upon *Lives* ought to be regulated, there being a great Difference between the Life of a Man of 20, and one of 50. For *Example*: 'Tis 100 to 1, that a Man of 20 dies not in a Year, and but 38 to 1, for a Man of 50 Years of Age. And upon these also depends the *Valuation of Annuities* for Lives; for it is plain, that the *Purchaser* ought to pay only such a Part of the *Value* of any *Annuity*, as he hath *Chances* that he is living.

And for that Purpose he hath taken the Pains (*which was not a little*) to compute the following *Table* (that shews the *Value of Annuities*) for every Fifth Year of Age to the 70th.

Age	Year's Purchase.	Age	Year's Purchase.	Age	Year's Purchase.
1	10,28	25	12,27	50	9,21
5	13,40	30	11,72	55	8,51
10	13,44	35	11,12	60	7,60
15	13,33	40	10,57	65	6,54
20	12,78	45	9,91	70	5,32

The same *ingenious Gentleman* proceeds on, and shews how to *estimate* or find the *Value of Two Lives*, and then of *Three Lives*, which being too long a Discourse to be recited here, I have, for Brevity's Sake, omitted it; and shall only add this serious Observation,

Viz. How unjustly we repine at the Shortness of our *Lives*, and think ourselves wrong'd if we attain not to *Old Age*; whereas it appears, that the *One Half* of those, that are BORN, die in Seventeen Years Time. For by the aforesaid Bills of Mortality at *Breslaw*, it was found, that 1238 were in that Time reduced to 616. So that, instead of murmuring at what we call a *Short Life*, we ought to account it as a great Blessing that we have *surviv'd*, perhaps by many Years, that *Period of Life* whereat the one Half of the whole Race of Mankind does not arrive.

Señ. 4. Of Purchasing Freehold, or Real Estates; at Compound Interest.

All *Freehold* or *Real Estates*, are supposed to be purchased or bought to continue for ever (*viz. without any limited Time*); therefore the Business of computing the true *Value* of such *Estates* is grounded upon a *Rank* or *Series* of *Geometrical Proportions* continually decreasing, *ad Infinitum*.

Thus, let P , u , R , denote the same *Data* as in the last *Section*. Then the *Series* will be, $\frac{u}{R}$, $\frac{u}{RR}$, $\frac{u}{R^3}$, $\frac{u}{R^4}$, $\frac{u}{R^5}$, and so on in $\div$ until the last *Term* = 0. Then will $P - 0$ (*viz.* P) be the sum of all the *Antecedents*. And $P - \frac{u}{R}$ will be the *Sum* of all the *Consequents*; therefore it will be $u : \frac{u}{R} :: P : P - \frac{u}{R}$ which produces $PR - u = P$.

This *Equation* affords the following *Theorems*.

Theorem 1. $PR - P = u$. *Theorem 2.* $\left\{ \frac{u}{R-1} = P \right.$

Theorem 3. $\left\{ \frac{P+u}{P} = R \right.$

Example. Suppose a *Freehold Estate* of 75 *l.* *Yearly Rent* were to be sold; what is it worth, allowing the *Buyer* 6 per Cent. &c. *Compound Interest* for his *Money*?

In this *Question* there is given $u = 75$. $R = 1,06$ to find P . Per *Theorem 2.* Thus $R - 1 = 0,06$ $75 = u$ ($1250 \text{ l.} = P$ the *Answer* required. And so on for any of the rest, as *Occasion* requires. But if the *Rent* is to be paid, either by *Quarterly* or *Half Yearly Payments*;

Then $R = \sqrt{1,06}$ for *Half Yearly* } *Payments at 6 per Cent.*
And $R = \sqrt[4]{1,06}$ for *Quarterly*

Or $\left\{ \begin{array}{l} R = 1,08 \text{ for Yearly} \\ R = \sqrt{1,08} \text{ for Half Yearly} \\ R = \sqrt[4]{1,08} \text{ for Quarterly} \end{array} \right\}$ *Payments at 8 per Cent.*

The like is to be understood for any other proposed *Rate* of *Interest*, either greater or less than 6 per Cent.

The *Application* of these *Theorems* to *Practice* is so very easy, that it's needless to insert more *Examples*.

AN

A N

INTRODUCTION

TO THE

Mathematicks.

P A R T III.

C H A P. I.

Of Geometrical Definitions, &c.

Seet. I. Of Lines and Angles.

A POINT *bath no Parts*: That is, a Geometrical Point is not any Quantity, but only an assignable Place in any Quantity, denoted by a Point: As } *A.* *B.*
at *A.* and *B.*

Such a Place may be conceived so infinitely small, as to be void of Length, Breadth, and Thickness; and therefore a Point may be said to have no Parts.

2. A LINE is called a Quantity of one Dimension, because it may have any supposed Length, but no Breadth nor Thickness, being made or represented to the Eye, by the Motion of a Point.

That is, if the Point at *A*, be moved (upon the same Plane) to the Point at *B*, it will describe a Line either right or circular (*viz.* *crooked*) according to its Motion.

Therefore the Ends or Limits of a Line are Points.

3. A RIGHT LINE, is that Line which lieth even or straight betwixt those Points that limit its Length, being the shortest Line that can be drawn between any Two } *A* ——— *B.*
Points. As the Line *AB.*

Therefore, between any two Points, there can lie or be drawn but one right Line.

4. A CIRCULAR, *crooked* or OBLIQUE Line, is that which lies bending between those Points which limit its Length, as the Lines CD or FG , &c.

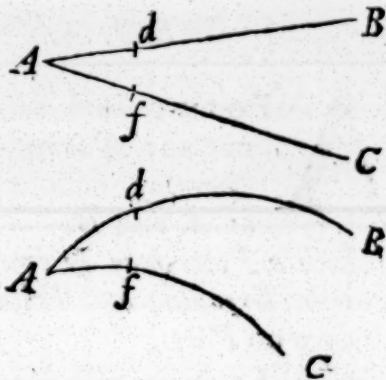
Of these Kinds of Lines there are various Sorts; but those of the Circle, Parabola, Ellipsis, and Hyperbola are of most general Use in Geometry; of which a particular Account shall be given further on.



5. PARALLEL LINES, are those that lie equally distant from one another in all their Parts, viz. such Lines as being infinitely extended (upon the same Plane) will never meet: As the Lines AB and ab : or CD and cd .



6. LINES not PARALLEL, but INCLINING (*viz. leaning*) one towards another, whether they are Right Lines, or Circular Lines, will (if they are extended) meet and make an Angle; the Point where they meet is called the Angular Point, as at A . And according as such Lines stand, nearer or further off each other, the Angle is said to be lesser or greater, whether the Lines that include the Angle be long or short. That is, the Lines Ad and Af include the same Angle as AB and AC doth; notwithstanding that AB is longer than Ad , &c.



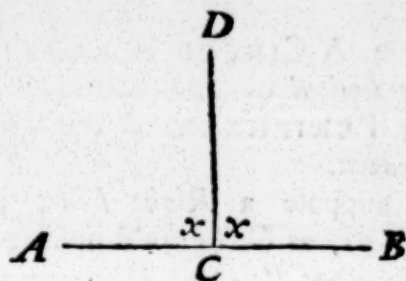
7. All ANGLES including between Right Lines are called Right-lin'd Angles; and those included between Circular Lines are called Spherical Angles. But all Angles, whether Right-lin'd or Spherical, fall under one of these Three Denominations.

Viz. $\left\{ \begin{array}{l} \text{A Right Angle.} \\ \text{An Obtuse Angle.} \\ \text{An Acute Angle.} \end{array} \right.$

8. A RIGHT-ANGLE is that which is included betwixt Two Lines, that meet one another Perpendicularly.

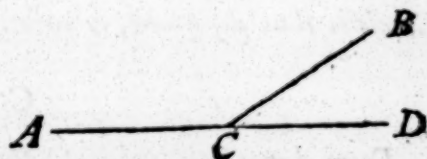
That

That is, when a *Right Line*, as *DC*, meets with another *Right-Line*, as *AB*, so directly as that it neither inclines nor declines to one *Side* more than the other, but make the *Angles* on both *Sides* of it equal, as at x, x ; then are those *Angles* called *Right-Angles*; and the *Lines* so meeting are said to be *Perpendicular* to each other.



That is, *AC*, and *CB*, are *Perpendicular* to *DC*, as well as *D, C* is to either or both of them.

9. An *OBTUSE ANGLE* is that which is greater than a *Right Angle*. Such is the *Angle* included between the *Lines AC* and *CB*.



10. An *ACUTE ANGLE* is that which is less than a *Right Angle*: As the *Angle* included between the *Lines CB* and *CD*.

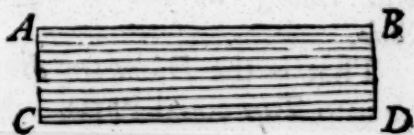
These *Two Angles* are generally called *OBLIQUE Angles*.

SECT. 2. Of a Circle, &c.

Before a *Circle* and its *Parts* are defined, it will be convenient to give a brief *Account* of *Superficies* in general.

1. A *SUPERFICIES* or *SURFACE* is the *Upper*, or very *Out-side* of any *visible Thing*. But by *Superficies* in *GEOMETRY*, is meant only so much of the *Out-side* of any *Thing* as is *inclosed* within a *Line* or *Lines*, according to the *Form* or *Figure* of the *Thing* designed; and it is *produced* or *formed* by the *Motion* of a *Line*, as a *Line* is described by the *Motion* of a *Point*; thus:

Suppose the *Line AB* were equally moved (upon the same *Plane*) to *CD*; then will the *Points* at *A* and *B* describe the two *Lines AC* and *BD*; and by so doing they



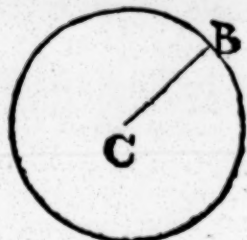
will form (and inclose) the *SUPERFICIES* or *Figure ABCD*, being a *Quantity* of *Two Dimensions*, viz. it hath *Length* and *Breadth*, but not *Thickness*. Consequently the *Bounds* or *Limits* of a *Superficies* are *Lines*.

Note,

Note, *The Superficies of any Figure, is usually called its AREA.*

2. A CIRCLE is a plain regular Figure, whose Area is bounded or limited by one continued Line, called the CIRCUMFERENCE or PERIPHERY of the Circle, which may be thus described or drawn.

Suppose a Right Line, as CB , to have one of its Extream Points, as C , so fix'd upon any Plane, as that the other Point at B may move about it; then if the Point at B be moved round about (upon the same Plane) it will describe a Line equally distant in all its Parts from the Point C , which will be the Circumference or Periphery of that Circle; the Point C will be its CENTER, and the contained Space will be its Area, and the Right Line CB , by which the Circle is thus described, is called RADIUS.



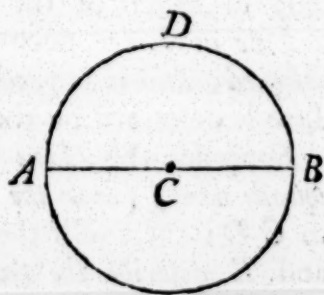
Confectary.

From hence 'tis evident, that an infinite Number of Right Lines may be drawn from the Center of any Circle to touch its Periphery, which will be all equal to one another, because they are all Radius's. And with a little Consideration it will be easy to conceive, that no more than two equal Right Lines can be drawn from any Point within a Circle to touch its Periphery, but from the Center only. (9. e. 3.)

3. EQUAL CIRCLES are those which have equal Radius's; for it's plain by the last Definition, that one and the same Radius (as CB) must needs describe equal Circles, how many soever they are.

4. The Diameter of a Circle, is twice its Radius joined into one Right Line; as AB drawn through the Center C , and ending at the Periphery on each Side.

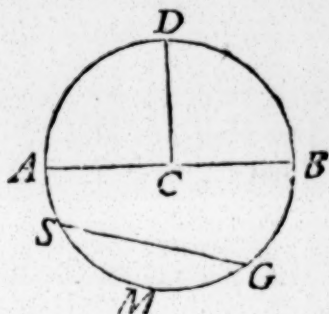
That is the Diameter divides the Circle into Two equal Parts.



5. A Semicircle (viz. Half a Circle) is a Figure included between the Diameter, and Half the Periphery cut off by the Diameter; as ADB .

6. A

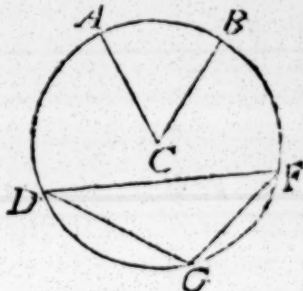
6. A QUADRANT is *Half a Semicircle*, viz. one *Quarter* of a Circle; and 'tis made by the *Radius* (as *D C*) standing *Perpendicular* upon the *Diameter* at the *Center C*, cutting the *Periphery* of the *Semicircle* in the *Middle*, as at *D*. Therefore a *Quadrant*, or *half the Semicircle*, is the *Measure of a Right Angle*.



7. A CHORD LINE, or the *Subtense* of an *Arch*, is any *Right Line* that cuts the *Circle* into *Two unequal Parts*, as the *Line S G*; and is always *less* than the *Diameter*.

8. A SEGMENT of a *Circle*, is a *Figure* included betwixt the *Chord* and that *Arch* of the *Periphery* which is cut off by the *Chord*: And it may either be *greater* or *less* than a *Semicircle*; as the *Figure S D G*, or *S M G*.

9. A SECTOR is a *Figure* included between *Two Radius's* of the *Circle*, and that *Arch* of its *Periphery* where they touch, as the *Figure A C B*: And the *Arch A B* is the *Measure* of the *Angle* at *C*, included betwixt the *Radius's A C* and *B C*.



Note, All *Angles* of *Sectors* are called *Angles* at the *Center* of a *Circle*.

10. AN ANGLE in the *Segment* of a *Circle* is that which is included between *Two Chords* that flow from one and the same *Point* in the *Periphery*, as at *D*, and meet with the *Ends* of another *Chord Line*, as at *F* and *G*.

That is, the *Angles* at *D*, at *F* and at *G*, are called *Angles* at the *Periphery*, or *Angles* standing on the *Segment* of a *Circle*.

SECT. 3. OF TRIANGLES.

There are two *Kinds* of *Triangles*, viz. *Plain* and *Spherical*; but I shall not give any *Definition* of the *Spherical*, because they more immediately relate to *Astronomy*.

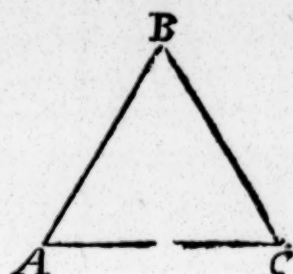
1. A PLAIN TRIANGLE is a *Figure* whose *Area* is contained within the *Limits* of *Three Right Lines* called *Sides*, including *Three Angles*: And it may be divided, and takes its *Name*, either according to its *Sides* or *Angles*.

1. By

I. By its SIDES.

2. An EQUILATERAL TRIANGLE is that which hath all its *Three Sides* equal; as the *Figure A B C*.

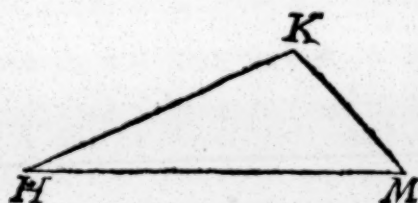
That is, $AB = BC = AC$.



3. An ISOSCELES TRIANGLE, is that which hath *only Two* of its *Sides* equal, as the *Figure B D G*: That is, $BD = DG$; but the *Third Side* BG may be either *greater* or *less*, as Occasion requires.

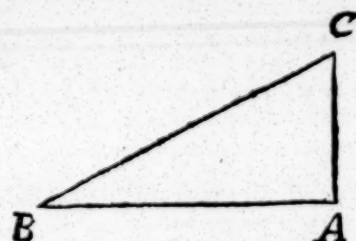


4. A SCALENE TRIANGLE, is that which hath all its *Three Sides* unequal; such as the *Figure H K M*.



2. By its ANGLES.

5. A RIGHT-ANGLED Triangle, is that which hath one *Right Angle*; that is, when *Two* of its *Sides* are *Perpendicular* to each other, as CA is supposed to be to BA . Therefore the *Angle* at A , is a *Right Angle*, per *Defin. 8. Sect. 1*.



Note, The *longest Side* of every *Right-angled Triangle* (as BC) is called the *Hypothenuse*, and the *longest* of the other *Two Sides* which include the *Right Angle* (as BA) is called the *Base*: The *Third Side* (as CA) is called the *Cathetus* or *Perpendicular*.

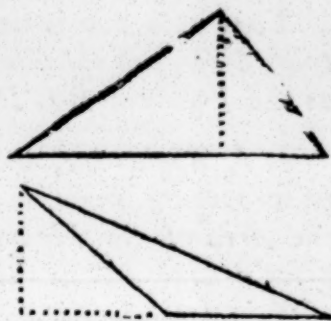
6. An OBTUSE-ANGLED Triangle, is that which hath *one* of its *Angles* *Obtuse*, and it's called an *Amblygonium Triangle*. Such is the *Third Triangle H K M*.

7. An ACUTE-ANGLED TRIANGLE, is that which hath all its *Angles* *Acute*, and it's called an *Oxygonium Triangle*; such are the *First* and *Second Triangles A B C* and *B D G*.

Note, All Triangles that have not a *Right Angle*, whether they are *Acute*, or *Obtuse*, are, in general Terms, called *Oblique Triangles*,

gles, without any other Distinction, as before. And the longest Side of every oblique Triangle is usually called the Base; the other two are only called Sides or Legs.

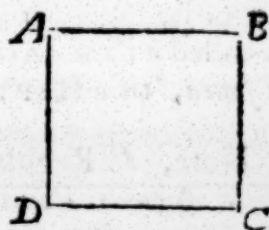
8. The ALTITUDE or HEIGHT of any Plain Triangle, is the Length of a Right Line let fall perpendicular from any of its Angles, upon the Side opposite to that Angle from whence it falls; and may be either within, or without the Triangle, as Occasion requires, being denoted by the Two prick'd Lines, in the annexed Triangles.



SECT. 4. Of Four-sided Figures.

1. A SQUARE is a plain regular Figure, whose Area is limited by Four equal Sides all perpendicular one to another.

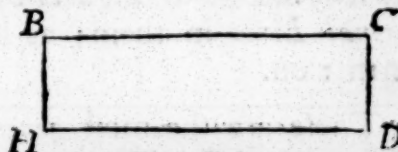
That is, when $AB = BC = CD = DA$, and the Angles A, B, C, D are all equal, then it's usually called a Geometrical Square.



2. A RHOMBUS, or Diamond-like Figure, is that which hath Four equal Sides, but no Right-angle. That is, a Rhombus is a Square mov'd out of its right Position, as the annexed Figure.



3. A RECTANGLE, or a Right-angled Parallelogram (often called an Oblong, or long Square) is a Figure that hath four Right-angles and its two opposite Sides equal, viz. $BC = HD$ and $BH = CD$.



4. A RHOMBOIDES, is an Oblique-angled Parallelogram; that is, it is a Parallelogram moved out of its right Position, like the annexed Figure.

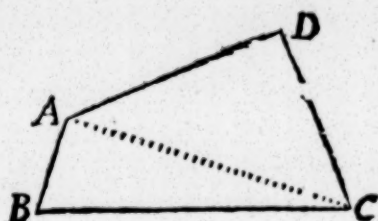


5. The ALTITUDE or Height of any Oblique-angled Parallelogram, viz. either of the Rhombus or Rhomboides, is a Right-line let fall perpendicular from any Angle upon the Side opposite to that Angle; and may either be within or without the Figure: As the prick'd Lines in the annexed Figure.



6. Every *Four-sided Figure*, different from those before-mentioned, is called a **TRAPEZIUM**.

That is, when it has neither *opposite Sides*, nor *opposite Angles equal*; as the *Figure A B C D*.



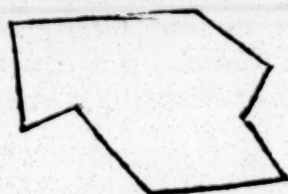
7. A *Right-line*, drawn from any *Angle* in a *Four-sided Figure* to its *opposite Angle*, is called a **DIAGONAL Line**, and will divide the *Area* of the *Figure* into two *Triangles*, being denoted by the prick'd Line *A C* in the last *Figure*.

8. All *Right lin'd Figures*, that have more than *four Sides*, are call'd **Polygons**, whether they be *regular* or *irregular*.

9. A **REGULAR POLYGON** is that which hath all its *Sides equal*, standing at *equal Angles*, and is named according to the *Number* of its *Sides* (or *Angles*). That is, if it have *five equal Sides*, it is called a **PENTAGON**; if *six equal Sides*, it is call'd a **HEXAGON**; if *seven*, 'tis a **HEPTAGON**; if *eight*, 'tis an **OCTAGON**, &c.

Note, All **Regular Polygons** may be *incrib'd* in a *Circle*; that is, their *Angular Points*, how many soever they have, will all just touch the *Circle's Periphery*.

10. An **IRREGULAR POLYGON** is that *Figure* which hath many *unequal Sides* standing at *unequal Angles* (like unto the *annexed Figure*, or otherwise); and of such Kind of *Polygons* there are *infinite Varieties*, but they may all be reduced to *regular Figures* by drawing *Diagonal Lines* in them; as shall be shew'd farther on.



These are the most *general* and *useful Definitions* that concern *plain* or *superficial Geometry*.

As for those which relate to *Solids*, I thought it convenient to omit giving any Account of them in this Place, because they would rather puzzle and amuse the *Learner*, than improve him, until he has gain'd a *competent Knowledge* in the most useful *Theorems* concerning *Superficies*; for then those *Definitions* may be more easily understood, and will help him to form a *clearer Idea* of their *respective Solids*, than 'tis possible to conceive of them before; and therefore I have reserv'd those *Definitions* until we come to the *Fifth Part*.

Sect. 5. Of such Terms as are generally used in Geometry.

Whatsoever is propos'd in Geometry will either be a PROBLEM or a THEOREM.

Both which Euclid includes in the general Term of Proposition.

A PROBLEM is that which *proposes* something to be done, and relates more immediately to *practical* than *speculative* Geometry; That is, it's generally of such a Nature, as to be performed by some known or *Commonly-receiv'd* Rules, without any Regard had to their *Inventions* or *Demonstrations*.

A THEOREM is when any *Commonly-receiv'd* Rule, or any *New Proposition* is required to be *demonstrated*, that so it may from thence forward become a *certain* Rule, to be *rely'd* upon in Practice when Occasion requires it. And therefore several Rules are often call'd *Theorems*, by which *Operations* in *Arithmetick*, and *Conclusions* in *Geometry*, are *perform'd*.

Note, By DEMONSTRATION is understood the highest Degree of Proof that human Reason is capable of attaining to, by a Train of Arguments deduced or drawn from such plain Axioms, and other Self-evident Truths, as cannot be denied by any one that considers them.

A COROLLARY, or CONSECTARY, is some Consequent Truth drawn or gain'd from any *Demonstration*.

A LEMMA is the *Demonstration* of some Premises laid down or propos'd as *preparative* to obviate and shorten the Proof of the Theorem under Consideration.

A SCHOLIUM is a brief Commentary or Observation made upon some precedent Discourse.

N. B. I advise the young Geometer to be very perfect in the Definitions, viz. Not to rest satisfied with a bare Remembrance of them; but, that he endeavour to gain a clear Idea or Understanding of the Things defined; and for that Reason I have been fuller in every Definition than is usual.

And, that he may know from whence most of the following Problems and Theorems contain'd in the Two next Chapters are collected, I have all along cited the Proposition and Book of Euclid's Elements where they may be found.

As for Instance; at Problem I. there is (3 e. 1.) which shews that it is the Third Proposition in Euclid's First Book. The like must be understood in the Theorems.

CHAP. II.

The First Rudiments, or Leading and Preparatory Problems, in Plain Geometry.

IN order to perform the following Problems, the young Geometer ought to be provided with a thin streight Ruler, made either of Brass or Box-wood, and two Pair of very good Compasses, viz. one Pair call'd Three-pointed Compasses, being very useful for drawing of Figures or Schemes, either with Black Lead or Ink; and one Pair of plain Compasses with very fine Points, to measure and set off Distances; also he should have a very good Steel Drawing Pen: And then he may proceed to the Work with this Caution; that he ought to make himself Master of one Problem before he undertakes the next: That is, he ought to understand the Design, and, as far as he can, the Reason of every Problem, as well as how to do it; and then a little Practice will render them very easy, they being all grounded upon these following Postulates.

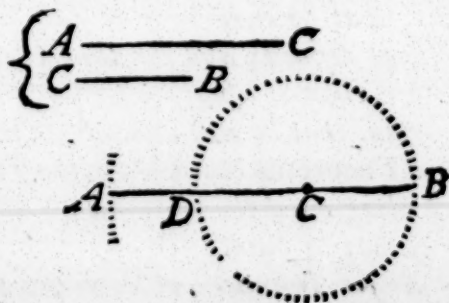
Postulates or Petitions.

1. That a Right-line may be drawn from any one given Point to another.
2. That a Right-line may be produced, increased, or made longer from either of its Ends.
3. That upon any given Point (or Center) and with any given Distance (viz. with any RADIUS) a Circle may be described.

PROBLEM I.

Two Right-lines being given, to find their Sum and Difference. (3. e. 1.)

Let the given Lines be



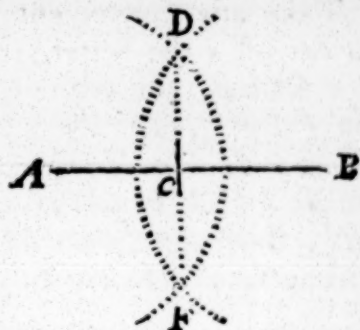
Make the shortest Line, as CB , Radius, and with it describe a Circle: From its Center C set off the other Line AC , and join ACB with a Right-line. Then will $AB = AC + CB$; and $AD = AC - CB$; as was required.

PRO.

PROBLEM II.

To bisect, or divide a Right-line given (as AB) into two equal Parts (10. e. 1.)

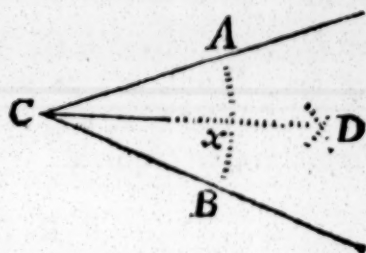
From both Ends of the given Line (*viz.* A and B) with any Radius greater than half its Length, describe Two Arches that may cross each other in two Points, as at D and F ; then join those Points $D F$ with a Right-line, and it will bisect the Line AB in the Middle at C ; *viz.* it will make $AC = CB$; as was required.



PROBLEM III.

To bisect a Right-lin'd Angle given, into two equal Angles. (9. e. 1.)

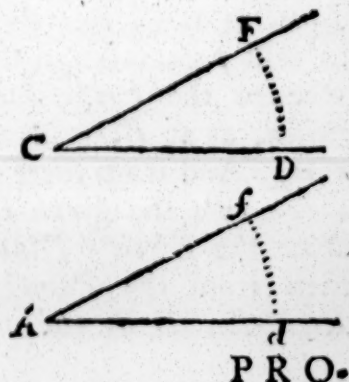
Upon the Angular Point, as at C , with any convenient Radius, describe an Arch as AB ; and from those Points A and B , describe two equal Arches crossing each other, as at D ; then join the Points C and D with a Right-line, and it will bisect the Arch AB , and consequently the Angle; as was requir'd.



PROBLEM IV.

At a Point A , in a Right-line given AB , to make a Right-lin'd Angle equal to a Right-lined Angle given C . (23. e. 1.)

Upon the given Angular Point C describe an Arch, as FD , (making CD any Radius at Pleasure) and with the same Radius describe the like Arch upon the given Point A , as fd ; that is, make the Arch fd equal to the Arch FD ; Then join the Points A and f with a Right-line, and it will form the Angle requir'd.

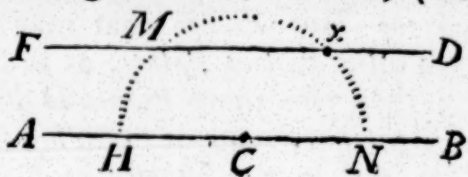


P R O.

PROBLEM V.

To draw a Right-line, as FD , parallel to a given Right-line AB , that shall pass thro' any assign'd Point, as at x , viz. at any Distance required. (31. e. 1.)

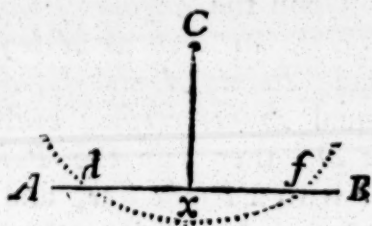
Take any convenient Point in the given Line, as at C , (the farther off x the better;) make Cx Radius, and with it upon the Point C , describe a Semi-circle, as HMN ; then make the Arch HM equal to the Arch xN ; thro' the Points M and x draw the Right-line FD , and it will be parallel to the Line AC , as was requir'd.



PROBLEM VI.

To let fall a Perpendicular, as Cx , upon a given Right-line AB , from any assign'd Point that is not in it, as from C . (12. e. 1.)

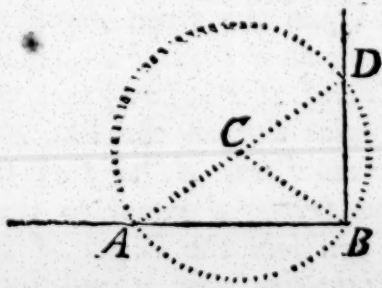
Upon the given Point C describe such an Arch of a Circle as will cross the given Line AB in two Points, as at d and f ; Then bisect the Distance between those two Points df (per Probl. 2.) as at x . Draw the Right-line Cx , and it will be the Perpendicular requir'd.



PROBLEM VII.

To erect or raise a Perpendicular upon the End of any given Right-line, as at B ; or upon any other Point assign'd in it. (11. e. 1.)

Upon any Point (taken at an Adventure) out of the given Line, as at C , describe such a Circle as will pass through the Point from whence the Perpendicular must be raised, as at B , (viz. make CB Radius): And from the Point where the Circle cuts the given Line, as at A , draw the Circle's Diameter ACD ; then from the Point D draw the Right-line DB , and it will be the Perpendicular as was requir'd.

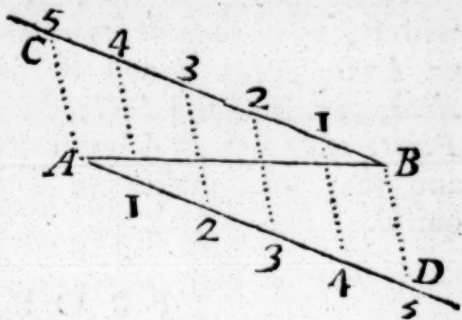


PRO.

P R O B L E M VIII.

To divide any given Right-line, as *AB*, into any proposed Number of equal Parts. (10. e. 6.)

At the extreame Points (or Ends) of the given Line, as at *A* and *B*, make two equal Angles (by Prob. 4.) continuing their Sides *AD* and *BC* to any sufficient Length; then upon those Sides, beginning at the Points *A* and *B*, set off the proposed Number of equal Parts (suppose 'em 5.) If Right-lines be drawn (cross the given Line) from one Point to the other, as in the annexed Figure, those Lines will divide the given Line *AB* into the Number of equal Parts required.

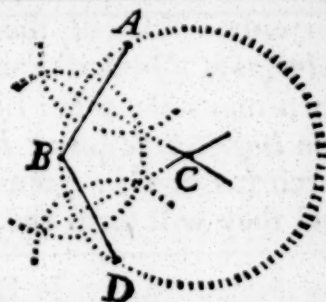


P R O B L E M IX.

To describe a Circle that shall pass (or cut) thro' any Three Points given, not lying in a Right-line, as at the Points *ABD*.

Join the Points *AB* and *BD* with Right-lines; then bisect both those Lines (per Problem 2.) the Point where the bisecting Lines meet, as at *C*, will be the Center of the Circle required.

The Work of this Problem being well understood, 'twill be easy to perform the two following, without any Scheme, viz.



1. To find the Center of any Circle given. (1. e. 3.)

By the last Problem 'tis plain, that if three Points be any where taken in the given Circle's Periphery, as at *A, B, D*, the Center of that Circle may be found as before.

2. If a Segment of any Circle be given, to compleat or describe the whole Circle.

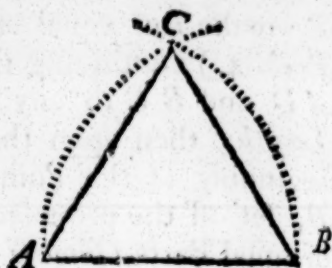
This may be done by taking any three Points in the given Segment's Arch, and then proceed as before.

P R Q.

P R O B L E M X.

Upon a Right-line given, as AB , to describe an Equilateral Triangle.
(1. e. 1.)

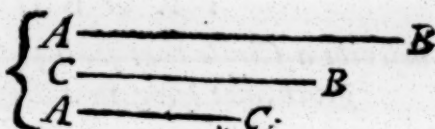
Make the given Line *Radius*, and with it, upon each of its *extream Points* or *Ends*, as at A and B , describe an Arch, viz. AC and BC ; then join the Points AC and BC with Right-lines, and they will make the Triangle requir'd.



P R O B L E M XI.

Three Right-lines being given, to form them into a Triangle, (provided any two of them, taken together, be longer than the Third)
(22. e. 1.)

Let the given Lines be

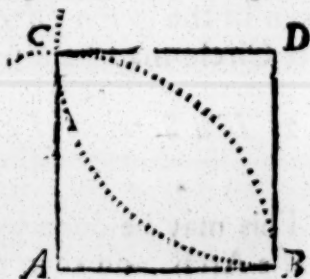


Make either of the *shorter Lines* (as AC) *Radius*, and upon either End of the *longest Line* (as at A) describe an Arch; then make the other Line CB *Radius*, and upon the other End of the *longest Side* (as at B) describe another Arch, to cross the First Arch (as at G): Join the Points CA and CB with Right-lines, and they will form the Triangle required.

P R O B L E M XII.

Upon a given Right-line, as AB , to form a Square. (46. e. 1.)

Upon one End of the given Line, as at B , erect the *Perpendicular* BD , equal in Length with the given Line, viz. make $BD = AB$; that being done, make the given Line *Radius*, and upon the Points A and D describe equal Arches to cross each other, as at C ; then join the Points CA and CD with Right-lines, and they will form the Square required.

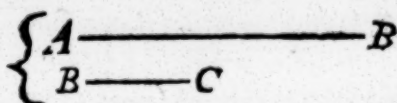


P R O.

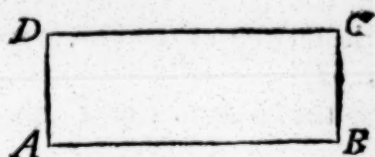
PROBLEM XIII.

Two unequal Right-lines being given, to form or make of them a Right-angled Parallelogram.

Let the given Lines be



Upon one End of the longest Line, as at B, erect a Perpendicular of the same Length with the shortest Line B C; then from the Point C draw a Line parallel and of the same Length, to A B; viz. make $DC = AB$: Join D A with a Right-line, and it will form the Oblong or Parallelogram required.



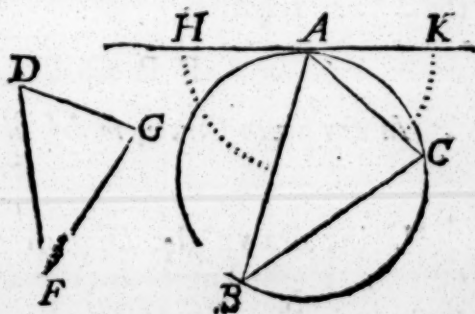
As for Rhombus's and Rhomboides's, to wit, Oblique-angled Parallelograms, they are made, or describ'd, after the same Manner with the two last Figures; only instead of erecting the Perpendiculars, you must set off their given Angles, and then proceed to draw their Sides parallel, &c. as before.

PROBLEM XIV.

In any given Circle, to inscribe or make a Triangle, whose Angles shall be equal to the Angles of a given Triangle; as the Triangle F D G, (2. e. 4.)

Note, Any Right-lin'd Figure is said to be inscrib'd in a Circle, when all the Angular Points of that Figure do just touch the Circle's Periphery.

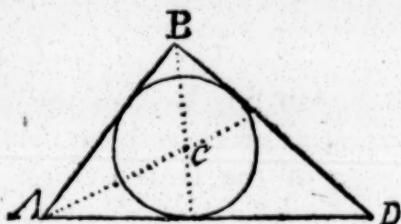
Draw any Right-line (as H K) so as just to touch the Circle, as at A; then make the Angle K A C equal to any one Angle of the given Triangle, as D F G; and the Angle H A B equal to another Angle of the Triangle, as D G F; then will the Angle B A C be equal to the Angle F D G. Join the Points B and C with a Right-line, and 'twill form the Triangle required.



P R O B L E M X V.

In any given Triangle, as $A B D$, to describe a Circle that shall touch all its Sides. (4. e. 4.)

Bisect any two Angles of the Triangle, as A and B , and where the bisecting Lines meet (as at C) will be the Center of the Circle required; and its Radius will be the nearest Distance to the Sides of the Triangle.



P R O B L E M X VI.

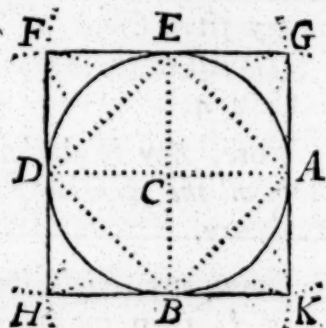
To describe a Circle about any given Triangle. (5. e. 4.)

This Problem is perform'd in all Respects like the Ninth, viz. by bisecting any Two Sides of the given Triangle; the Point, where those bisecting Lines meet, will be the Center of the Circle required.

P R O B L E M X VII.

To describe a Square about any given Circle. (7. e. 4.)

Draw two Diameters in the given Circle (as $D A$ and $E B$) crossing at Right Angles in the Center C ; and, with the Circle's Radius $C A$, describe from the extrem Points of those Diameters, viz. A , B , D , E , cross Arches, as at F , G , H , K ; then join those Points where the Arches cross with Right-lines, and they will form the Square required.



P R O B L E M X VIII.

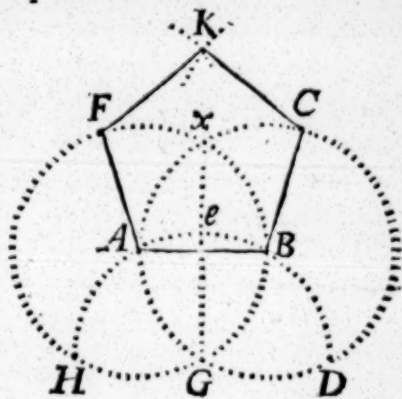
In any given Circle, to describe the largest Square it can contain. (6. e. 4.)

Having drawn the Diameters, as $D A$ and $E B$, bisecting each other at Right-angles in the Center C , (as in the last Scheme); then join the Points A , B , D , and E , with Right-lines, viz. $A B$, $B D$, $D E$, $E A$, and they will be Sides of the Square required.

PROBLEM XIX.

Upon any given Right line, as AB , to describe a regular Pentagon, or Five-sided Polygon.

Make the given Line *Radius*, and upon each End of it describe a Circle; and through those Points where the Circles cross each other (as at G & x) draw the Right-line Gex : Upon the Point G with the same *Radius* describe the Arch $HAeBD$, and laying a Ruler upon the Points D, e , mark where it crosses the other Circle, as at F . Again, lay the Ruler upon the Points H, e , and mark where it crosses the other Circle, as at C : Then from the Points F and C (with the same *Radius* as before) describe cross Arches, as at K : Join the Points AF, FK, KC , and CB , with Right-lines, and they will form the *Pentagon* required, viz. $AF = FK = KC = CB = AB$; and the Angles at A, B, C, K, F will be equal.

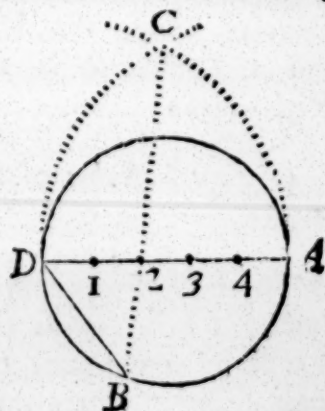


PROBLEM XX.

In any given Circle, to describe a regular Pentagon.
(II. e. 4. & IO. e. 3.)

Or, in general Terms, to describe any regular Polygon in a Circle.

Draw the Circle's Diameter DA , and divide it into as many equal Parts as the proposed *Polygon* hath Sides; then make the whole Diameter a *Radius*, and describe the two Arches CA and CD . If a Right line be drawn from the Point C , through the Second of those equal Parts in the Diameter, as at 2, it will assign a Point in the opposite Semicircle's *Periphery*, as at B . Join DB with a Right-line, and it will be the Side of the *Pentagon* required.



Qq 2

These

These Twenty Problems are sufficient to exercise the young Practitioner, and bring his Hand to the right Management of a Ruler and Compasses, wherein I would advise him to be very ready and exact.

As to the Reason why such Lines must be so drawn as directed at each Problem, that, I presume, will fully and clearly appear from the following Theorems; and therefore I have (*for Brevity's Sake*) omitted giving any Demonstrations of them in this Chapter, desiring the Learner to be satisfied with the bare Knowledge of doing them only, until he hath fully considered the Contents of the next Chapter; and then I doubt not but all will appear very plain and easy.

CHAP. III.

A Collection of most useful Theorems in plain Geometry Demonstrated.

Note, In order to shorten several of the following Demonstrations, it will be necessary to premise, that

1. **T**HE Periphery (or Circumference) of every Circle (*whether great or small*) is suppos'd to be divided into 360 equal Parts, called *Degrees*; and every one of those *Degrees* are divided into 60 equal Parts, called *Minutes*, &c.

2. All *Angles* are measured by the Arch of a Circle describ'd upon the *Angular Point* (*See Defin. 9. Page 287.*) and are esteem'd greater or less, according to the Number of *Degrees* contain'd in that *Arch*.

3. A *Quadrant*, or *Quarter-part* of any Circle, is always 90 *Degrees*, being the *Measure* of a *Right-angle* (*Defin. 6. P. 287.*) and a *Semicircle* is 180 *Degrees*, being the *Measure* of two *Right-angles*.

4. *Equal Arches* of a Circle, or of equal Circles, measure equal *Angles*.

To those five general *Axioms* already laid down in Page 146, (*which I here suppose the Reader to be very well acquainted with*) it will be convenient to understand these following, which begin their Number where the other ended.

Axioms.

Axioms.

6. Every whole Thing is GREATER than its PART.

That is, the whole Line AB is } greater than its Part Ac , &c. } $A \text{ ————— } |^c \text{ ————— } B$
 The same is to be understood of *Superficies's* and *Solids*.

7. Every Whole is EQUAL to all its PARTS taken together.

That is, the whole Line AB is equal } to its Parts $AC + cd + de + eB$. } $A \text{ — } |^c \text{ — } |^d \text{ — } |^e \text{ — } B$
 The same is also true in *Superficies's* and *Solids*.

8. Those Things which being laid one upon another, do agree or meet in all their Parts, are equal one to the other.

But the Converse of this *Axiom*, to wit, that equal Things being laid one upon the other will meet, is only true in *Lines* and *Angles*, but not in *Superficies's*, unless they be alike, viz. of the same *Figure* or *Form*: As for Instance, a Circle may be equal in *Area* to a Square; but if they are laid one upon the other, 'tis plain they cannot meet in all their Parts, because they are unlike *Figures*. Also, a *Parallelogram* and a *Triangle* may be equal in their *Area's* one to another, and both of them may be equal to a Square; but if they are laid one upon the other, they will not meet in all their Parts, &c.

Note, Besides the Characters already explain'd in Part I, and in other Places of this *Treat*, these following are added.

Viz. $\sphericalangle$ denotes an Angle in general, and $\sphericalangle \sphericalangle$ signifies Angles; $\triangle$ signifies a Triangle; $\square$ signifies a Square, and $\square$ denotes a Parallelogram. And when an Angle is denoted by any three Letters (as, ABC) the middle Letter (as B) always denotes the Angular Point; and the other two Letters (as AB and BC) denote the Lines or Sides of the Triangle which includes that Angle.

These Things being premised, the young Geometer may proceed to the *Demonstrations* of the following *Theorems*; wherein he may perceive an absolute Necessity of being well versed in several Things that have been already deliver'd: And also it will be very advantageous to store up several useful *Corollaries* and *Lemma's*, as they become discover'd Truths: For it often happens, that a *Proposition* cannot be clearly demonstrated *a priori*, or of itself, without a great Deal of Trouble; therefore it will be useful to have Recourse to those Truths that may be assisting in, the *Demonstrations* then in Hand.

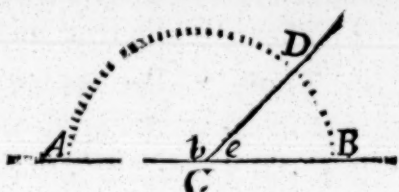
THEO-

THEOREM I.

If a Right-line stand upon (or meet with) another Right-line, and make Angles with it, they will either be two Right-angles, or two Angles equal to two Right-angles. (13. e. 1.)

Demonstration.

Suppose the Lines to be AB and DC , meeting in the Point at C : Upon C describe any Circle at pleasure; then will the Arch AD be the Measure of the $\angle b$, and the Arch DB the Measure of $\angle e$; but the Arches $AD + DB = 180^\circ$, viz. they complet the Semicircle.



Consequently the $\angle b + \angle e = 180^\circ$. Which was to be prov'd.

Corollaries.

I. Hence it follows, that if the $\angle b = 90^\circ$ then $\angle e = 90^\circ$; but if $\angle b$ be obtuse, then the $\angle e$ will be acute, &c.

From hence it will be easy to conceive, that if several Right-lines stand upon, or meet with any Right-line at one and the same Point, and on the same Side, then all the Angles taken together will be $= 180^\circ$, viz. Two Right-angles.

THEOREM II.

If two Angles intersect (i. e. cut or cross) each other, the two opposite Angles will be equal. (15. e. 1.)

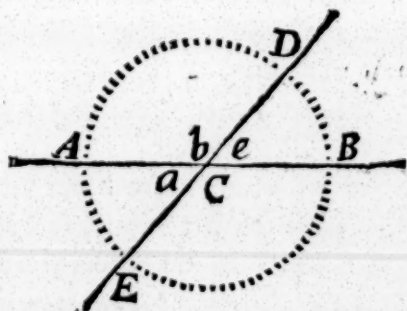
Demonstration.

Let the two Lines be AB and DE , intersecting each other in the Center C .

Then $\angle b + \angle e = 180^\circ$
And $\angle b + \angle a = 180^\circ$ } per last.
Consequently $\angle b + \angle e = \angle b + \angle a$, per Axiom 5.

Subtract $\angle b$ on both Sides of the Equation, and it will leave $\angle e = \angle a$.

Again, $\angle b + \angle e = 180^\circ$, as before; and $\angle e + \angle c = 180^\circ$, consequently $\angle e + \angle c = \angle b + \angle e$. Subtract $\angle e$, and then $\angle c = \angle b$. Q. E. D.



Corol.

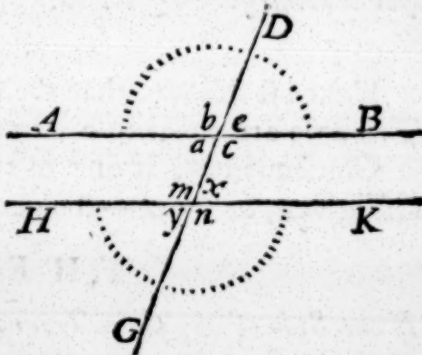
Corollary.

From hence it is evident, that if two Lines intersect each other, they will make four Angles; which, being taken together, will always be equal to Four Right-angles.

THEOREM III.

If a Right-line cut (or cross) two parallel Lines, it will make the opposite Angles equal one to another. (29. e. 1.)

Suppose the two Lines AB and HK to be parallel, and the Right line DG to cut them both at C and n : Upon the Point C (with any Radius) describe a Semicircle; and with the same Radius, upon the Point at n , describe another Semicircle opposite to the first, as in the Figure. Then 'tis plain, and I suppose very easy to conceive, that if the Center C were mov'd along upon the Line DG , until it came to the Center at n , the two Lines AB and HK would meet and concur, viz. become one Line (for parallel Lines are as it were but one broad Line). Consequently the two Semicircles would also meet, and become one entire Circle, like to that in the last Demonstration.



And therefore the $\angle y = \angle x = \angle a = \angle e$ } as before, per
 And $\angle m = \angle n = \angle b = \angle c$ } last Theorem.
 Q. E. D.

Corollary.

Hence it follows, that if three, four, or ever so many Parallel-lines, are cut or cross'd by one Right-line, all their opposite Angles will be equal.

THEOREM IV.

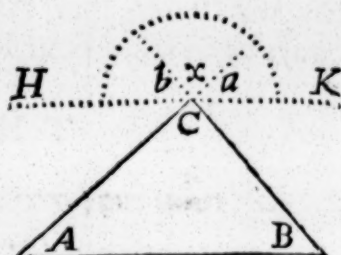
The three Angles of every plain Triangle are equal to two Right-angles. (32. e. 1.)

Consequently, any two Angles of any plain Triangle must needs be less than two Right-angles. (17. e. 1.)

Demon

Demonstration.

Let the $\triangle ABC$ be propos'd; draw the Right-line HK parallel to the Side AB , just touching the Vertical Angle C ; and upon the same Angular Point C describe any Semicircle, and produce the Sides AC and BC to its Periphery. Then will $\sphericalangle b = \sphericalangle B$, $\sphericalangle a = \sphericalangle A$, and $\sphericalangle x = \sphericalangle C$, per last Theorem. But $\sphericalangle b + \sphericalangle a + \sphericalangle x = 180^\circ$, or two Right-angles: Consequently $\sphericalangle B + \sphericalangle A + \sphericalangle C = 180^\circ$ per Axiom 5. Q. E. D.



Corollary.

Hence it follows, that the two acute Angles of every Right-angled Triangle are equal to a Right-angle, or 90° .

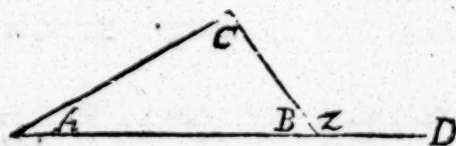
Consequently, if one of the acute Angles be given, the other is also given, viz. 90° —the given $\sphericalangle$ leaves the other $\sphericalangle$.

THEOREM V.

If one Side of any plain Triangle be continued or produced beyond, or out of the Triangle, the outward Angle will always be equal to the two inward opposite Angles. (32. e. 1.)

Demonstration.

Let the Side AB of the $\triangle ABC$ be produced out of the $\triangle$, suppose to D , &c. as in the Figure. Then $\sphericalangle z = \sphericalangle A + \sphericalangle C$, for the $\sphericalangle B + \sphericalangle z = 180^\circ$ per Theorem 1. and the $\sphericalangle B + \sphericalangle A + \sphericalangle C = 180^\circ$, per last Theorem. Therefore $\sphericalangle B + \sphericalangle z = \sphericalangle B + \sphericalangle A + \sphericalangle C$, per Axiom 5. Subtract $\sphericalangle B$ on both Sides the Equation, and it will leave $\sphericalangle z = \sphericalangle A + \sphericalangle C$ (per Axiom 2.) Q. E. D.



Consequently, the outward Angle (at z) of any plain Triangle, must needs be greater than either of the inward opposite Angles, viz. greater than $\sphericalangle A$, or $\sphericalangle C$ (16. e. 1.)

Corollary.

Hence it follows, that if one Angle of any plain Triangle be given, the Sum of the other two Angles is also given; for 180° —the given $\sphericalangle =$ the other two $\sphericalangle$'s.

THEO.

THEOREM VI.

In every plain Triangle, equal Sides subtend (viz. are opposite to) equal Angles. (5. e. 1.)

Consequently, equal Angles are subtended by equal Sides (6. e. 1.)

Demonstration.

Suppose the $\triangle BCD$ to be an *Isoceles* $\triangle$; that is, let $BC = CD$. Bisect the $\angle C$, or (which is all one) make CA perpendicular to BD ; then will the $\angle$ on each Side of it (viz. $\angle BAC$ and $\angle DAC$) be Right-angles.



Therefore $\left\{ \begin{array}{l} \frac{1}{2} \angle C + \angle B = 90^\circ \\ \frac{1}{2} \angle C + \angle D = 90^\circ \end{array} \right\}$ per Corol. to Theorem 4.

Consequently, $\frac{1}{2} \angle C + \angle B = \frac{1}{2} \angle C + \angle D$, per Axiom 5. Subtract $\frac{1}{2} \angle C$ from both Sides of the Equation, and it will leave $\angle B = \angle D$, per Axiom 2. Q. E. D.

Corollary.

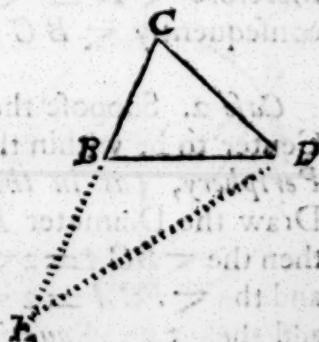
From hence it follows, that the three Angles of an Equilateral Triangle are equal one to another.

THEOREM VII.

In every plain Triangle, the longest Side subtends the greatest Angle. (18. e. 1.)

Consequently, the greatest Angle of any plain Triangle is subtended by the longest Side.

This Theorem is evident by Inspection only: For, let one of the Sides of any plain Triangle (as CB) be produced, suppose to E ; join DE with a Right-line; then 'tis evident, that because CE is now made longer than the Side BC , therefore the $\angle$ at D is become larger than it was before by the $\angle BDE$: And it's plain, the longer the Side CE had been made, the $\angle$ at D would have been the more enlarged.



THEOREM VIII.

If the Sides of two Triangles are equal, the Angles opposite to those equal Sides will be equal. (8. e. 1.)

The Truth of this Theorem is evident by the two included Triangles in the 6th Theorem, for they have their respective Sides equal, viz. $BC = CD$, $BA = DA$, and CA common to both Triangles. And it is there prov'd, that the $\angle$ opposite to those equal Sides are equal, &c. which needs no further Proof.

Note, The Converse of this Theorem holds not true; for the Angles of two Triangles may be equal, and their opposite or subtending Sides unequal; as will appear at Theorem XII.

Corollary.

Hence it follows, that Triangles mutually equilateral are also mutually equiangular; and,

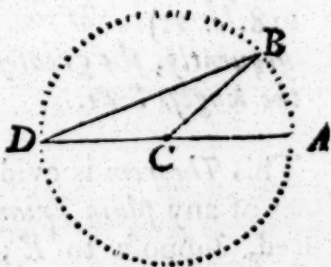
That Triangles mutually equilateral are equal one to another. (4. & 26. e. 1.)

THEOREM IX.

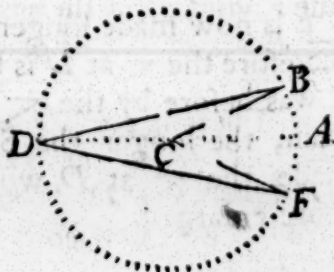
An Angle at the Center of any Circle is always double to the Angle at the Periphery, when both the Angles stand upon the same Arch. (20. e. 3.) This Theorem hath three Varieties or Cases.

Demonstration.

Case 1. Let the Diameter DA , and the Line DB , be the two Lines which form the $\angle D$ at the Periphery; draw the Radius BC , then $\angle BCA$ is the $\angle$ at the Center. But $\angle BCA = \angle D + \angle B$, per Th. 5. and because $DC = BC$, therefore $\angle D = \angle B$, per Theorem 6. consequently $\angle BCA = 2 \angle D$.



Case 2. Suppose the $\angle BCF$ at the Center to be within the $\angle BDF$ at the Periphery, (as in the annexed Figure.) Draw the Diameter DA ; then the $\angle BCA = 2 \angle BDA$ and the $\angle FCA = 2 \angle FDA$ } per Case 1. add these two Equations together.



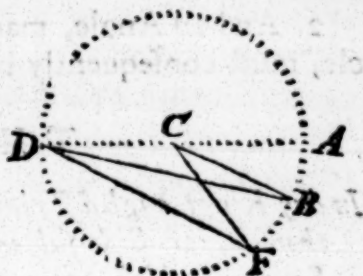
Then

Then will $\angle BCA + \angle FCA = 2 \angle BDA + 2 \angle FDA$, per Ax. 1. But $\angle BCA + \angle FCA = \angle BCF$, and $2 \angle BDA + 2 \angle FDA = 2 \angle BDF$. Consequently $\angle BCF = 2 \angle BDF$.

Case 3. Again, suppose the $\angle BCF$ at the Center to be out of the $\angle BDF$ at the Periphery. From the Angular Point D at the Periphery draw the Diameter DA .

Then $\angle FCA = 2 \angle FDA$ } per Case 1.
and $\angle BCA = 2 \angle BDA$ }

Subtract this last Equation from the other, and it will leave $\angle FCA - \angle BCA = 2 \angle FDA - 2 \angle BDA$, per Axiom 2. But $\angle FCA - \angle BCA = \angle FCB$, and $2 \angle FDA - 2 \angle BDA = 2 \angle FDB$: Consequently $\angle FCB = 2 \angle FDE$. Q. E. D.



Corollary.

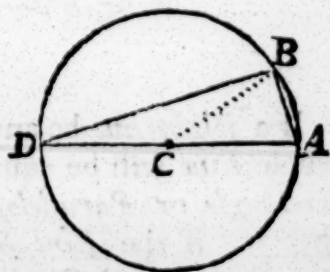
Hence 'tis evident, that all Angles at the Periphery, which stand on the same Segment or Arch of a Circle, or upon equal Arches, are equal one to another. (21. e. 3.)

THEOREM X.

An Angle in a Semicircle is a Right-angle. (31. e. 3.) That is, if the Diameter of any Circle be the Side of a Triangle, and the Angle opposite to that Side be any where in the Circle's Periphery, it will be a Right-angle.

Demonstration.

Let DA be the Diameter, and DBA the Triangle, then, $\angle B = 90^\circ$. Draw the Radius BC , then is the $\angle DBA = \angle D + \angle A$. For $\angle CBD = \angle D$, and $\angle CBA = \angle A$, per Theorem 6. Therefore $\angle DBA = \angle CBD + \angle CBA$, per Axiom 5. Again $\angle DBA + \angle D + \angle A = 180^\circ$, per Theorem 4. Consequently, $\angle DBA = 90^\circ$ or a Right-angle. Q. E. D.



Corollaries.

1. Hence it will be easy to conceive, that an Angle made in any Segment less than a Semicircle will be *obtuse*, or greater than a Right-angle.

2. And an Angle, made in any Segment greater than a Semicircle, must consequently be *acute*.

THEOREM XI.

In any Right-angled Triangle, the Square which is made of the Hypotenuse, or Side subtending the Right-angle, is equal to both the Squares which are made of the Sides including the Right-angle. (47. e. 1.)

There are several Ways of demonstrating this noble and useful Theorem, but, I presume, none more easy to be understood by a Learner than that which I shall here propose: And, in order thereto, 'twill be necessary to premise the following Lemma's.

Lemma 1.

A Right-line is said to be multiply'd with a Right-line, when either a Square, or other Right-angled Parallelogram, is made of the two Lines.

That is, the Area of any Right-angled Parallelogram is equal to the Product of those Numbers which express the Measure of its Sides.

Thus, if $AB = 6$ Inches and $AC = 3$ Inches: Then $AB \times AC = 6 \times 3 = 18$ square Inches; which is the Area of the Parallelogram $ABCD$.



Lemma 2.

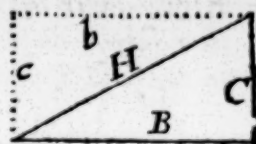
If a Right-line be any way cut into two Parts, the Square of the whole Line will be equal to the Squares of each Part, and a double Rectangle or Parallelogram made of both the Parts, (4. e. 2.) that is, if the Line S be cut into the two Parts B and C ; then is $S = B + C$:
 But if both the Sides of the Equation be involv'd, it will be $SS = BB + 2BC + CC$.

Lemma

Lemma 3.

The *Area* of every *Right-angled Triangle* is half the *Parallelogram* made of its *Base* and *Perpendicular*.

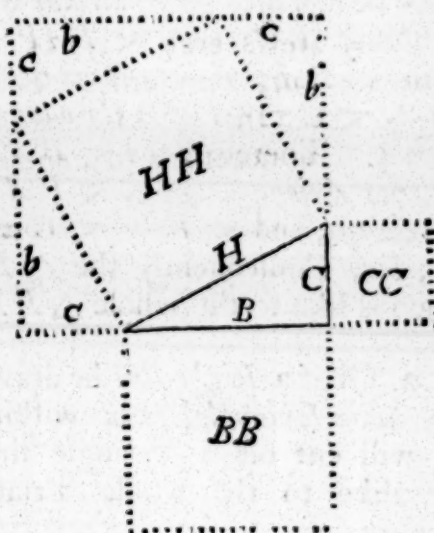
For $B \times C =$ the *Area* of the whole *Parallelogram*, by the first *Lemma*. And $\triangle B C H + \triangle b c H =$ the *Parallelogram*; but $B = b$, and $C = c$. Therefore $\frac{1}{2} B \times C =$ the *Area* of each $\triangle$, viz. $\frac{1}{2} B \times C + \frac{1}{2} b \times c = B \times C$.



These Things being premised, let us suppose the *Triangle* $B C H$ to be a *Right-angled Triangle*, viz. the *Side* C perpendicular to the *Side* B ; then will $B B + C C = H H$.

Demonstration.

Make a *Square* whose *Side* is $= B + C$, and draw the included *Square* whose *Side* is $= H$, as in the *Scheme*: Then will the *Area* of the great *Square* be equal to the *Area* of the four *Triangles* $+ H H$; but the *Area* of each $\triangle = \frac{1}{2} B C$, or $B \times C$, per *Lemma 3*. Therefore the 4 $\triangle$'s $= \frac{1}{2} B C \times 4 = 2 B C$, consequently, the *Area* of the great *Square* is $H H + 2 B C$. Involve $B + C$, and it will be $B B + 2 B C + C C =$ the *Area* of the great *Square*; per *Lemma 3*. Consequently, $H H + 2 B C = B B + 2 B C + C C$, per *Axiom 5*. Subtract $2 B C$ from both *Sides* of the *Equation*, and there will remain $H H = B B + C C$.



To illustrate this *Theorem* by *Numbers*, let us

Suppose $C = 3$. $B = 4$. and $H = 5$.

Then will $C C = 9$. $B B = 16$. and $H H = 25$.

Consequently, $B B + C C = H H = 16 + 9 = 25$.

Conseſſary.

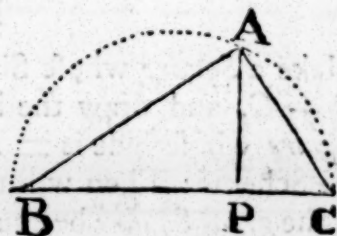
From this admirable *Theorem* (said to be first invented by *Pythagoras*) is deduced the *Method* of adding and subtracting *Squares*, *Parallelograms*, *Circles*, &c.

THEOREM XII.

In any Right-angled Triangle, a Perpendicular being let fall from the Right-angle upon the Hypotenuse will divide the Triangle into two Right-angled Triangles, which will be both similar (or alike) to the first Triangle, and to each other. (8. e. 6.)

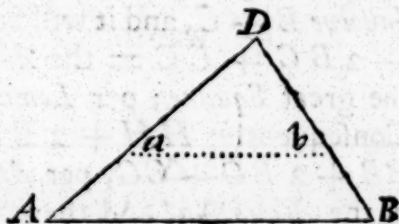
Note, All plain Triangles are said to be similar (viz. alike) when each single Angle in one of the Triangles is equal to each single Angle of the other; but if any two single Angles of one Triangle are equal to two single Angles of the other, the third Angle will be equal. Per Theo. 4.

1. In the Right-angled $\triangle BAC$, let AP be supposed perpendicular to the Hypotenuse BC ; then $\angle BAP = \angle C$. For $\angle BAP + \angle B = 90^\circ$, and $\angle B + \angle C = 90^\circ$, per Corollary to Theorem 4. Therefore the $\angle BAP = \angle C$, per Axiom 5. again, $\angle PAC + \angle C = 90^\circ$, and $\angle B + \angle C = 90^\circ$. Therefore $\angle PAC = \angle B$, &c. Consequently the $\triangle BAP$ is alike to the $\triangle ACP$; and each is like to the whole $\triangle BAC$.



2. Or if a Right-line be drawn parallel to one of the Sides of any plain Triangle, (viz. within it) it will cut off a Triangle similar or alike to the whole Triangle. Thus:

In the $\triangle ABD$ draw the Right-line ab parallel to the Side AB ; then will the included $\triangle aDb$ be like the $\triangle ADB$: For $\angle a = \angle A$ and $\angle b = \angle B$, per Theorem 3; and $\angle D$ is common to both the Triangles; Ergo, &c.



THEOREM XIII.

If two Triangles are alike, their like Sides will be proportional.

That is, those Sides which subtend the equal Angles, as also those Sides which are about the equal Angles, will be proportional to each other; and consequently, if any two Triangles have their Sides proportional, their Angles are equal. (4, 5, 6, 7. e. 6.)

Demon

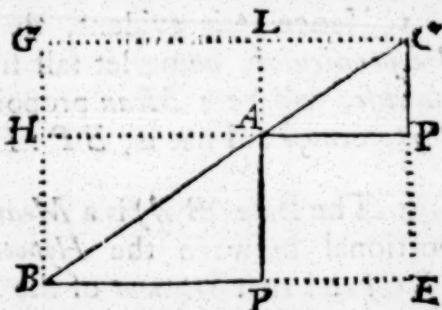
Demonstration.

Let the *similar Triangles* in the *Scheme* of the last *Theorem* be here proposed again.

Then it will be $BP : AP :: AP : CP$, according to this *Theorem*. Ergo $BP \times CP = AP \times AP$.

First.

Let us suppose the aforefaid *Right-angled* $\triangle BAC$ cut through the *Perpendicular* AP , and there open'd until the Sides BA and CA become one *Right-line*. Let the Sides BP and CP be continued until they meet in E ; then compleat the *Parallelograms* by drawing the parallel Lines GLC , HAP , GHB , and LAP , as in the *Figure*.

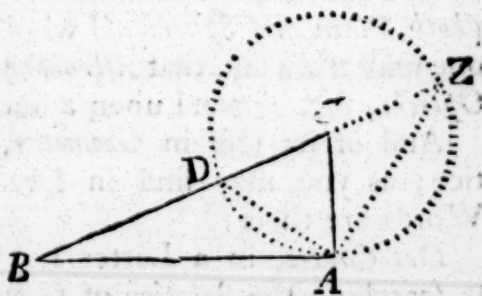


Then it is evident, that the $\triangle BHA = \triangle BPA$, and the $\triangle CPA = \triangle CLA$; also that the $\triangle BEC = \triangle BGC$, because all their respective Sides are *equal*.

But the $\triangle BHA + \triangle CLA + \square HGLA = \triangle BPA + \triangle CPA + \square APPE$. Now if from both *Sides* of this *Equation* there be subtracted the equal *Triangles*, there will remain $\square HGLA = \square APPE$. But $\square HGLA = BP \times CP$, and $\square APPE = AP \times AP$. Consequently $BP : AP :: AP : CP$. Which was to be prov'd.

Or otherwise, thus:

Suppose the $\triangle BAC$ to be *Right-angled* at A : Upon the $\angle$ Point C , with the *Radius* CA describe a *Circle*, and continue the *Hypothenuse* BC to Z ; join ZA and AD with *Right-lines*; then will the $\triangle BAD$ be like to the $\triangle BZA$. For $\angle DAB + \angle DAC = 90^\circ$, by *Construction*. And $\angle ZAC + \angle DAC = 90^\circ$, by *Theorem X*. Therefore $\angle DAB + \angle DAC = \angle ZAC + \angle DAC$. By *Axiom 5*. subtract $\angle DAC$ from both *Sides* of the *Equation*, and there will remain $\angle DAB = \angle ZAC$. But $\angle ZAC = \angle CZA$, by *Theorem 6*. And $\angle B$ is common



to both Triangles. Therefore $\angle BDA = \angle BAZ$, by Theorem 6, consequently $\triangle BAD$ is like to $\triangle BZA$.

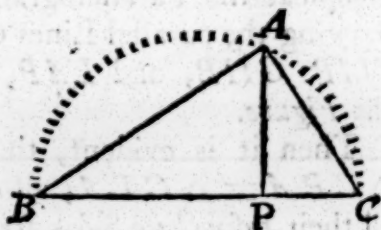
Let the Sides $\left\{ \begin{array}{l} BA = b \\ BC = h \\ CA = c \end{array} \right\}$ Then $bb + cc = hh$, by Theorem 11.
 Consequently $bb = hh = cc$,
 which gives the following Analogy,
 Viz. $b : b + c :: b - c : b$; that is, $BA : BZ :: BD : BA$.
 Q. E. D.

Corollaries.

1. Hence it is evident, that, in any *Right-angled Triangle*, a *Perpendicular*, being let fall from the *Right-angle* upon the *Hypotenuse*, will be a *Mean* proportional between the *Segments* of the *Hypotenuse*: That is, $BP : PA :: PA : PC$.

2. The *Base* (BA) is a *Mean* proportional between the *Hypotenuse* (BC) and that *Segment* of the *Hypotenuse* next to the *Base*, (viz. BP) that is, $BC : BA :: BA : BP$.

3. The *Cathetus* (AC) is a *Mean* proportional between the *Hypotenuse* (BC) and that *Segment* of the *Hypotenuse* next to the *Cathetus*, (viz. PC): That is, $BC : AC :: AC : PC$.



Scholium.

I have been more large upon this most excellent *Theorem*, in giving a double *Demonstration* of it, because it is so *universally useful* in all Parts of the *Mathematicks*: For the *Business* of *Trigonometry* (both *Plain* and *Spherical*) wholly depends upon it; and therefore one may truly say, that *Astronomy*, *Dialling*, *Navigation*, *Surveying*, *Opticks*, &c. depend upon a due Application of it.

And of its Use in *Geometry*, *Des-Cartes* takes particular Notice; as you may find in *Dr. Pell's Algebra*, *Pag. 65*, whose Words are these:

Des Cartes, in a Letter not yet printed, writes thus: " In
 " searching the *Solution* of *Geometrical Questions*, I always make
 " use of Lines parallel and perpendicular, as much as is possible,
 " [he means as many Lines as are useful] and I consider no
 " other *Theorems* but these two, [the Sides of like Triangles
 " have like Proportion]. And [in Rectangle Triangles
 " the

“ the Square of the greatest Side is equal to the Squares of the two other Sides.] And I am not afraid to suppose many unknown Quantities, that I may reduce the propos’d Question to such Terms, as to depend on no other Theorems but these Two.”

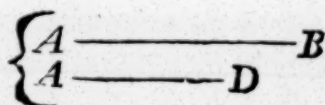
This I thought convenient to insert, that the young *Learner* may see how the great *Des-Cartes* esteem’d these two Theorems, viz. the last, and Theorem 11; for, in Truth, all the precedent Theorems are only (as it were) Preparatives to these Two.

This last Theorem demonstrates the *Reason* of the *Method* used in finding out *Proportional Lines*; as in the Three following *Problems*.

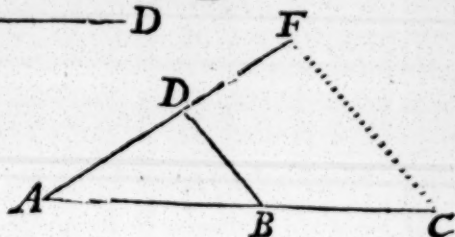
PROBLEM I.

Two Right-lines being given to find a Third in Proportion to them.
(11. e. 6.)

Let these two Lines be



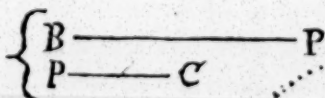
Set the Two given Lines at any Angle in the Point *A*, and produce the Line *AB* to *C*, making $BC = AD$; join the Points *B* *D* with a Right-line, and draw *CF* parallel to *BD*; then will the $\triangle ABD$ be like the $\triangle ACF$. Therefore $AB : BC (= AD) :: AD : DF$, which is the third Proportional requir’d.



PROBLEM II.

Two Right-lines being given, to find a Mean proportional Line between them. (13. e. 6.)

Let the given Lines be



Join the two given Lines into one, viz. make $BC = BP + PC$, and upon *BC*, as Diameter, describe a Semicircle; then upon the Point *P*, where the two Lines meet, erect a Perpendicular to touch the



S s

Cir-

Circle's *Periphery*, as PA , and it will be the *Mean proportional* requir'd, viz. $BP : AP :: AP : PC$.

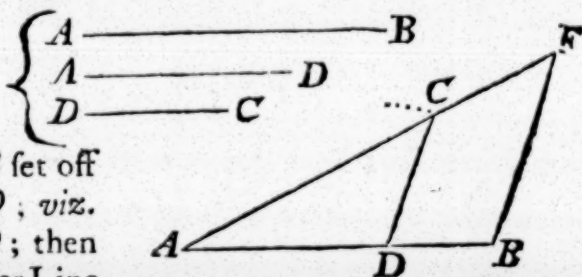
By this *Problem* 'tis easy to conceive how to make a *Square* equal to any given *Parallelogram*. (14. e. 6.)

For if BP be the *Length*, and PC be the *Breadth* of the given *Parallelogram*, then will AP be the *Side* of the *Square*, equal in *Area* to that *Parallelogram*.

PROBLEM III.

Three *Right-lines* being given, to find a fourth *Proportional Line*. (12. e. 6.)

Suppose the three Lines



Upon the longest Line AB set off the next longest Line AD ; viz. make $DB = AB - AD$; then upon the Point D set the other Line

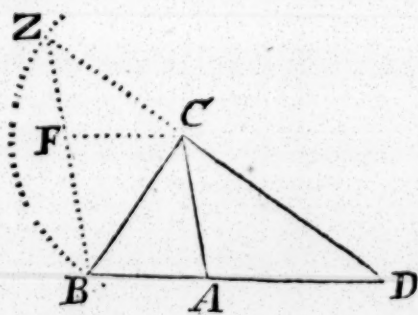
DC at an *Angle*, either *right* or *oblique*, and draw the *Right line* AC continuing it a sufficient Length; make BF parallel to DC , and it will be the fourth *Proportional* requir'd; that is, $AD : DC :: AB : BF$.

THEOREM XIV.

If any *Angle* of a plain *Triangle* be bisected (viz. divided into two equal *Angles*) with a *Right-line*, (viz. as CA is suppos'd to do the *Angle* BCD) it will cut the opposite *Side* (viz. BD) in *Proportion* to the other two *Sides* of the *Triangle* (3. e. 6.) i. e. $BA : BC :: AD : CD$.

Demonstration.

Produce the Side DC , until $CZ = CB$: join the Points ZB with a *Right-line*, and draw the Line FC parallel to BD ; whence the $\angle Z = \angle CBZ$ per *Theorem* 6. and $\angle Z + \angle CBZ$, or $2 \angle CBZ = \angle BCD$, per *Theorem* 5; or, dividing both Sides of the *Æquation* by 2, $\angle CBZ = \frac{1}{2} \angle BCD$. But $\frac{1}{2} \angle B$



$CD = \angle ACB = \angle ACD$ by the *Hypothesis*, therefore $\angle ACB = \angle CBZ$ per *Axiom* 5: Whence AC is parallel to BZ per *Theorem* 3, and the *Triangles* BDZ , ADC , and FCZ are similar by the second Figure to *Theorem* 12. consequently $BA (= FC) : BC (= ZC) :: AD : CD$. Q. E. D.

THEO-

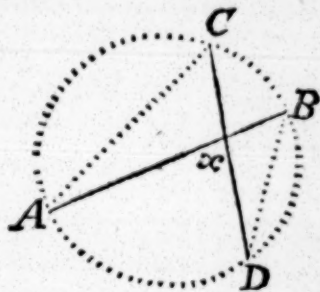
THEOREM XV.

If two Right-lines (howsoever drawn) within a Circle do cut each other, the Rectangle made of the Segments (or Parts) of the one Line, will be equal to the Rectangle made of the Segments (or Parts) of the other Line. (35. e. 3.)

That is, if two Lines (as AB and CD) do cut each other in any Point, as at x , then will $Ax \times Bx = Dx \times Cx$.

Demonstration.

Join the Points of AC and BD with Right-lines, then will the $\triangle Cx A$ be like to $\triangle Bx D$: For $\sphericalangle B = \sphericalangle C$ and $\sphericalangle A = \sphericalangle D$. by Corollary to Theorem 9. and $\sphericalangle AxC = \sphericalangle BxD$. by Theorem 2. Therefore it will be $Ax : D :: Cx : Bx$. by Theorem 13. Consequently $Ax \times Bx = Dx \times Cx$. Q. E. D.



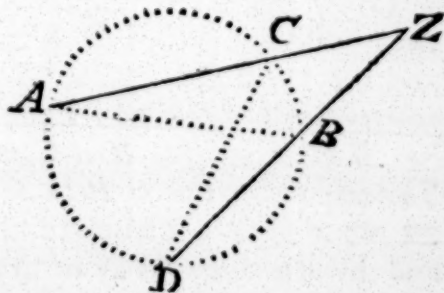
THEOREM XVI.

If two Right-Lines are so drawn within a Circle, as, being continued, they will meet in a Point out of the Circle's Periphery, the Rectangle made of the one whole Line, and its Part out of the Circle, will be equal to the Rectangle of the other whole Line, and its Part out of the Circle. (36, 37. e. 3.)

That is, if the Lines AC and DB be continued unto the Point Z ; then will $AZ \times CZ = DZ \times BZ$.

Demonstration.

Draw the Lines AB and CD , then will $\triangle CZD$ be like to the $\triangle BZA$; for $\sphericalangle A = \sphericalangle D$, and $\sphericalangle Z$ is common to both Triangles, consequently, $\sphericalangle ABZ = \sphericalangle DCZ$, by Theorem 4. therefore $AZ : BZ :: DZ : CZ$. Ergo, $AZ \times CZ = DZ \times BZ$.



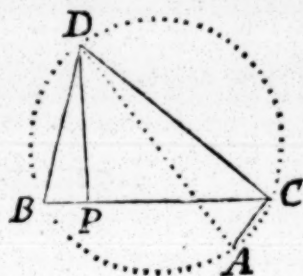
THEOREM XVII.

If from any Angle of a plain Triangle inscribed in a Circle there be let fall a Perpendicular upon the opposite Side, as DP ; as that Perpendicular

dicular is in Proportion to one of the Sides including the Angle, so is the other Side including the Angle to the Diameter of the Circle.

Demonstration.

Let BCD be the proposed Triangle. From the $\sphericalangle$ at D draw the Diameter DA ; then will $\sphericalangle A = \sphericalangle B$, because they both stand upon the same Arch DC , and $\sphericalangle DCA = 90^\circ$, by Theorem 10. consequently the $\sphericalangle ADC = \sphericalangle BDP$, by Theorem 4. Therefore $\triangle DCA$ is like to the $\triangle DPB$; and therefore, $DP : DB :: DC : DA$; or $DP : DC :: DB : DA$. Q. E. D.



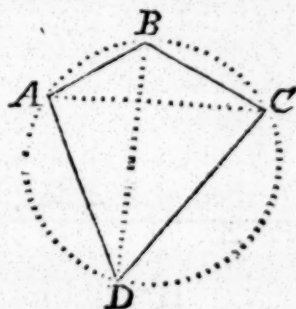
THEOREM XVIII.

If any Quadrangle (that is, a Trapezium) be inscrib'd within a Circle, the two opposite Angles, taken together, are equal to two Right-Angles, viz. 180° (22. e. 3.)

That is, in the Quadrangle $ABCD$ the $\sphericalangle A + \sphericalangle C = 180^\circ$. And the $\sphericalangle B + \sphericalangle D = 180^\circ$.

Demonstration.

Draw the two Diagonals AC and BD ; then will the $\sphericalangle BDA = \sphericalangle BCA$, and the $\sphericalangle BDC = \sphericalangle BAC$ by Corollary to Theorem 9. But $\sphericalangle ABC + \sphericalangle BCA + \sphericalangle BAC = 180^\circ$, by Theorem 4. and the $\sphericalangle BDA + \sphericalangle BDC = \sphericalangle ADC$. Therefore the $\sphericalangle ABC + \sphericalangle ADC = 180^\circ$, and by the same Way of arguing it may be prov'd, that the $\sphericalangle BAD + \sphericalangle BCD = 180^\circ$. Q. E. D.



THEOREM XIX.

If in any Quadrangle inscrib'd within a Circle there be drawn two Diagonals, as AC and BD , the Rectangle made of the two Diagonals will be equal to both the Rectangles made of the opposite Side of the Quadrangle.

That is, $AC \times BD = AB \times CD + AD \times BC$.

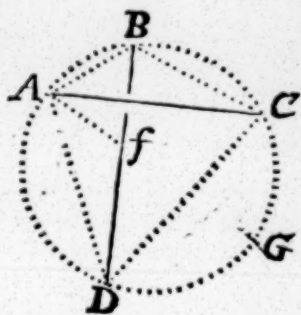
Demon

Demonstration.

Make the Arch $DG = \text{Arch } BC$, and from the Points A, G draw the Line Af , and it will form the $\triangle AfD$, like to the $\triangle ABC$: For the $\angle fAD = \angle BAC$, because the Arches DG and BC are equal.

Again, the $\angle fDA = \angle BCA$, because they both stand upon the Arch AB : Consequently, the $\angle AfD = \angle ABC$, by Theorem 4. Therefore it will be $AC : BC :: AD : Df$, by Theorem 13. Ergo $\frac{BC \times AD}{AC} = Df$.

Again, the $\triangle B Af$ and $\triangle ACD$ are alike: For $\angle ABf = \angle ACD$, and $\angle B Af = \angle CAD$, because the $\angle fAD = \angle BAC$, and the $\angle CAf$ is common to both Triangles. Consequently, the $\angle AfB = \angle ADC$. Therefore $AC : CD :: AB : Bf$, by Theorem 13. Ergo $\frac{CD \times AB}{AC} = Bf$. But $Df + Bf = BD$. Consequently, $BC \times AD + CD \times AB = BD \times AC$. Q. E. D.



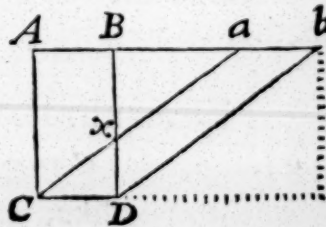
THEOREM XX.

All Parallelograms (whether Right or Oblique angled) that stand upon the same Base, or upon equal Bases, and betwixt the same Parallels, are equal to one another. (35. & 36. e. 1.)

That is, $\square ABCD = \square abCD$.

Demonstration.

Because $AB = CD = ab$, by Supposition, therefore $Aa = Bb$; for Ba is common to both. And because $AC = BD$, and the $\angle A = \angle B$, therefore the $\triangle ACa = \triangle BDb$: And if from both Triangles there be taken the $\triangle Bxa$ common to both, there will remain the Trapezium $ABxC = abxD$, per Axiom 5.



But

But the Trapezium $AB \times C + \triangle C \times D = \square ABCD$. and the Trapezium $ab \times D + \triangle C \times D = \square abCD$. consequently $\square ABCD = \square abCD$. Q. E. D.

Corollary.

Hence it will be easy to conceive, that all *Triangles* which stand upon the same Base, or upon equal Bases, and between the same Parallels, (viz. *having the same Height*) are equal one to another. (37 & 38 e. 1.)

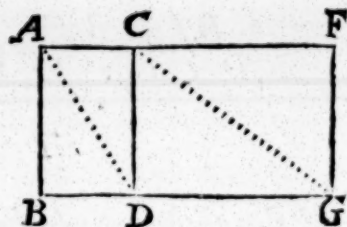
For all *Triangles* are the Halves of their circumscribing *Parallelograms*; and therefore, if the *Wholes* be equal, their *Halves* will also be equal.

T H E O R E M XXI.

Parallelograms (and consequently Triangles) which have the same Height, have the same Proportion one to another as their Bases have. (1. e. 6.)

Demonstration.

Draw AF parallel to BG , and draw AB, CD, FG Perpendiculars to them. Then will $BD \times AB = \square ABCD$. And because $CD = AB$, therefore $DG \times AB = \square CDFG$, but $BD : DG :: BD \times AB : DG \times AB$. And consequently $\triangle ABD : \triangle CDG :: BD : DG$, &c.



Q. E. D.

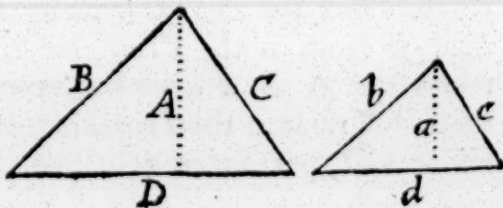
T H E O R E M XXII.

Like Triangles are in a duplicate Ratio to that of their homologous Sides. (19. e. 6.)

That is, the *Area's* of like *Triangles* are in *Proportion* one to another as are the *Squares* of their like *Sides*.

Demonstration.

Suppose the $\triangle BCD$ and $\triangle bcd$ to be alike, and their like Sides to be those mark'd with the same Letters,



Let

Let A and a be Perpendiculars to the two Bases D and d .
 Then $\frac{1}{2} DA =$ the Area of $\triangle BCD$ } By Lemma 3, Page 303.
 And $\frac{1}{2} da =$ the Area of $\triangle bcd$ }
 But 1 $B : b :: D : d$ } &c. By Theorem 13.
 And 2 $B : b :: A : a$ }
 Conseq. 3 $D : d :: A : a$
 3 $\therefore$ 4 $Da = dA$
 4 $\times \frac{1}{2} Dd$ 5 $\frac{1}{2} DDda = \frac{1}{2} DddA$. By Axiom 3.
 5, Hence 6 $DD : dd :: \frac{1}{2} DA : \frac{1}{2} da$. And so for other Sides.
 Q. E. D.

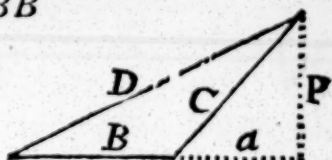
THEOREM XXIII.

In every Obtuse-angled Triangle (as BCD) the Square of the Side subtending the obtuse Angle (as D) is greater than the Squares of the other two Sides (B and C) by a double Rectangle made out of one of the Sides (as B) and the Segment or Part of that Side produced (as a) until it meet with the Perpendicular (P) let fall upon it. (12 e. 2.)

That is, $DD = BB + CC + 2Ba$.

Demonstration.

First 1 $DD = PP + aa + 2Ba + BB$
 And 2 $CC = PP + aa$
 1 — 2 3 $DD - CC = 2Ba + BB$
 1 + CC 4 $DD = BB + CC + 2Ba$



Q. E. D.

Corollary.

Hence it is evident, that, if the Sides of any Obtuse-angled Triangle are given, the Segment (a) of the Side produced (or the Perpendicular P) may be easily found.

THEOREM XXIV.

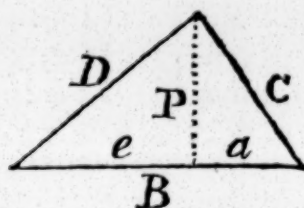
If a Perpendicular (as P) be let fall into any Acute-angled Triangle (as BCD), the Square of either of the Two Sides (as D) is less than the Squares of the other Side, and that Side upon which the Perpendicular falls (viz. C and B) by a double Rectangle made of the Side B , and that Segment or Part of it (viz. a) which lies next to the Side C . (13. e. 2.)

That is, $DD + 2Ba = BB + CC$.

Demon.

Demonstration.

First	1	$DD = PP + ee$	} By <i>Theo.</i> II.
And	2	$CC = PP + aa$	
But	3	$B - a = e$, by Figure.	
3	2	$BB - 2 Ba + aa = ee$.	
4	5	$BB - 2 Ba = ee - aa$.	
1	2	$DD - CC = ee - aa$.	
5,	6	$DD - CC = BB - 2 Ba$.	
7	±	$DD + 2 Ba = BB + CC$.	



Q. E. D.

Corollary.

Hence it follows, that, if the Sides of any *Acute-angled Triangle* be known, the Perpendicular *P*, and the Segments of the Side whereon it falls (*viz.* *a*, *e*.) may be easily found.

CHAP. IV.

The Solution of several Easy Problems in plain Geometry, whereby the Learner may (in Part) perceive the Application or Use of the foregoing Theorems.

NOTE, when a Line, or the Side of any plain Triangle, is any Way cut into two or more Parts, either by a Perpendicular Line let fall upon it, or otherwise, those Parts are usually call'd Segments; and so much as one of those Parts is longer than the other, is call'd the Difference of the Segments.

And when any Side of a Triangle, or any Segment of its Side is given, 'tis usually mark'd with a small Line cross it, thus: —|— and those Sides or Parts of Sides, that are sought, are marked with four Points, thus: —∴—

PROBLEM I.

To cut or divide a given Right line (as *S*) into Extreme and Mean Proportion. (II. e. 2.)

That is, to divide a Line so, that the Square of the greater Segment (or Part) *a*, may be equal to the Rectangle made of the whole Line *S*, and the lesser Segment *e*.

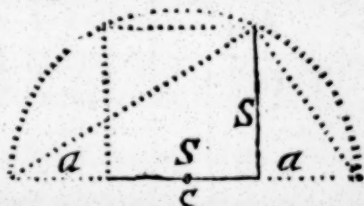
Viz. | 1 | $Se = aa$, by the Problem.
And | 2 | $S - a = e$, for $S = a + e$.

$$\begin{array}{r} S \\ \hline a \quad | \quad e \\ \hline 1 \div S \end{array}$$

1	÷	S	3	$\frac{aa}{S} = e$
2	and	3	4	$\frac{aa}{S} = S - a$. By Axiom 5.
4	×	S	5	$aa = SS - Sa$
5	+	Sa	6	$aa + Sa = SS$
6, solved			7	$a = \sqrt{SS + \frac{1}{2}SS} : \frac{1}{2}SS$. See Pages 195, 196.

Note, The last Problem cannot be truly answered by Numbers, but Geometrically it may be performed, thus :

1. Make a Square whose Side is = S the given Line, and bisect one of its Sides in the Middle, as at C ; upon the Point C describe such a Semicircle as will pass through the remotest Points of the Square, and compleat its Diameter.



2. Then will either Part of the Diameter, on each End of the Side S , be = a , the greater Segment sought.

But $a + S : S :: S : a$. By Theorem 13.

Ergo, $aa + Sa = SS$. Which was to be done.

PROBLEM II.

The Base of any Right-angled Triangle, and the Difference between the Hypotenuse and Cathetus being given, to find the Cathetus, &c.

Let {	1	$b = 72$
	2	$d = 32$
And	3	$a = \text{Cathetus sought}$

Then	4	$bb + aa = dd + 2da + aa$
		By Theorem 11.

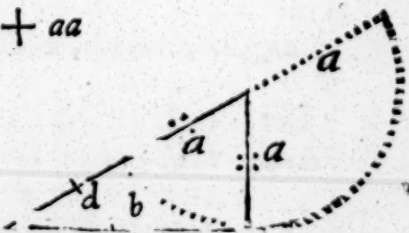
$$4 \text{ — } aa \quad 5 \quad bb = dd + 2da$$

$$5 \text{ — } dd \quad 6 \quad 2da = bb - dd$$

$$6 \div 2d \quad 7 \quad a = \frac{bb - dd}{2d} = 65$$

$$\text{Or, } 8 \quad b : d + 2a :: d : b. \text{ By Theorem 13.}$$

$$8 \quad \therefore 9 \quad bb = dd + 2da. \text{ As before at the 5th Step.}$$



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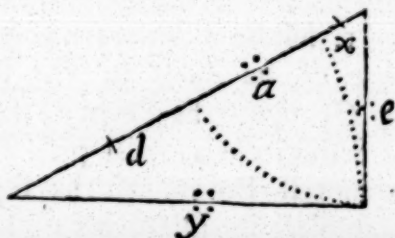
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Here you see that either Way raises the same *Equation*; neither is there any constant *Method* or *Road* to be observed in solving *Geometrical Problems*, but every one makes Use of such Ways and Theorems as happen to come first into their Mind, the Result being every Way the same.

PROBLEM III.

The Difference between the Base and Hypotenuse of any Right-angled Triangle, and the Difference between the Cathetus and Hypotenuse being both given, to find the Triangle.

Let {	1	$d = 32$
	2	$x = 25$
And	3	$d + x + a = \text{the Hypot.}$
Then {	4	$d + a = y$
	5	$x + a = e$
		} by the Probl.
4 $\ominus^2$	6	$dd + 2da + aa = yy$
5 $\ominus^2$	7	$xx + 2xa + aa = ee$
3 $\ominus^2$	8	$dd + 2dx + 2da + 2xa + xx + aa = \square \text{ Hypotenuse.}$
6 + 7	9	$dd + 2da + 2xa + xx + 2aa = yy + ee.$



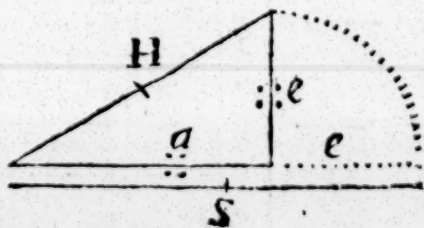
The two last Steps are equal, by *Theorem II*. Consequently, if those Things that are equal in both be taken away, the Remainders will be equal. By *Axiom 2*.

That is	10	$aa = 2dx = 1600$
10 ω^2	11	$a = \sqrt{2dx} = 40$
1 + 11	12	$d + a = 72 = y \text{ The Base.}$
2 + 11	13	$x + a = 65 = e \text{ The Cathetus.}$
1 + 2 + 11	14	$d + x + a = 97 \text{ The Hypotenuse.}$

PROBLEM IV.

The Hypotenuse, and the Sum of the other two Sides, of any Right-angled Triangle, being given, thence to find the Sides.

Let	1	$H = 97$
And	2	$a + e = S = 137$
By Fig.	3	$aa + ee = HH$
2 $\ominus^2$	4	$aa + 2ae + ee = SS$
4 — 3	5	$2ae = SS - HH$
3 — 5	6	$aa - 2ae + ee = 2HH - SS$
6 ω^2	7	$a - e = \sqrt{2HH - SS}$



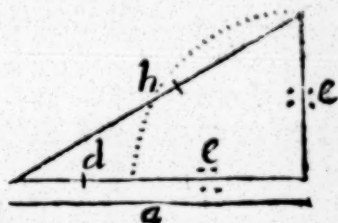
2 + 7

$$\begin{array}{r|l}
 2 + 7 & 8 \quad 2a = S + \sqrt{2HH - SS} = 144 \\
 8 + 2 & 9 \quad a = \frac{S + \sqrt{2HH - SS}}{2} = 72 \text{ The Base required.} \\
 2 - 9 & 10 \quad e = \frac{S - \sqrt{2HH - SS}}{2} = 65 \text{ The Cathetus.}
 \end{array}$$

PROBLEM V.

The Hypothenufe, and the Difference of the other two Sides of any Right-angled Triangle being given, to find the Sides.

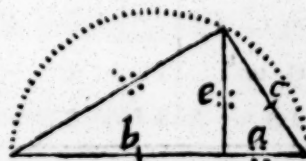
$$\begin{array}{r|l}
 \text{Let} & 1 \quad b = 97 \text{ As before.} \\
 \text{And} & 2 \quad a - e = d = 7 \text{ Quere } a \\
 \text{By Fig.} & 3 \quad aa + ee = bb \\
 2 \quad \odot^2 & 4 \quad aa - 2ae + ee = dd \\
 3 - 4 & 5 \quad 2ae = bb - dd \\
 3 + 5 & 6 \quad aa + 2ae + ee = 2bb - dd \\
 6 \quad w^2 & 7 \quad a + e = \sqrt{2bb - dd} \\
 2 + 7 & 8 \quad 2a = d + \sqrt{2bb - dd} = 144 \\
 8 \div 2 & 9 \quad a = 72 \\
 7 - 2 & 10 \quad 2e = \sqrt{2bb - dd} - d = 130 \\
 1 \div 2 & 11 \quad e = 65
 \end{array}$$



PROBLEM VI.

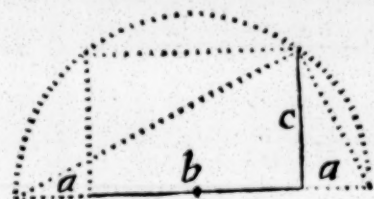
In any Right-angled Triangle, either the Base, or Cathetus, and the alternate Segment of the Hypothenufe made by a Perpendicular let fall from the Right-angle, being given, to find the other Segment.

$$\begin{array}{r|l}
 \text{Let} & 1 \quad e = 45 \text{ The Cathetus} \\
 \text{And} & 2 \quad b = 48 \text{ The alternate Segm.} \\
 \text{Then} & 3 \quad b : e :: e : a \text{ Quere } a \\
 3 & 4 \quad ba = ee \\
 \text{Again,} & 5 \quad cc - aa = ee. \text{ By Theor. II.} \\
 4, & 6 \quad ba = cc - aa \\
 6 + aa & 7 \quad aa + ba = cc \\
 7, C \square & 8 \quad aa + ba + \frac{1}{4}bb = cc + \frac{1}{4}bb \\
 8 \quad w^2 & 9 \quad a + \frac{1}{2}b = \sqrt{cc + \frac{1}{4}bb} \\
 9 - \frac{1}{2}b & 10 \quad a = \sqrt{cc + \frac{1}{4}bb} - \frac{1}{2}b = 27 \text{ And so on for } e, \&c.
 \end{array}$$



I shall now shew the *Geometrical Construction* (or *Solution*) of the three Cases of *Quadratick Equations* promised in 202. Let the first Example be that above, viz. $aa + ba = cc$. Case 1.

Make the Co-efficient b , and the Root of the *Resolvend* (which is here) c , into a *Right-angled Parallelogram*. And upon the middle Point of the Side $= b$, describe such a Semicircle, as will pass through the remotest Points or Angles of the *Parallelogram*, completing its Diameter, as in the annexed Scheme.

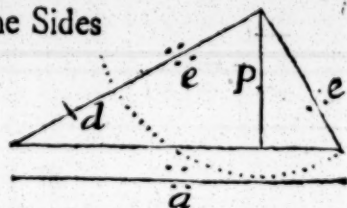


Then will either Part of the Diameter, on each End, be equal to a ; the other Part will be $a + b$, and the Side c will be a *mean Proportional* between them: That is, $a + b : c :: c : a$. By Theorem 13, consequently $aa + ba = cc$. Which was to be done.

PROBLEM VII.

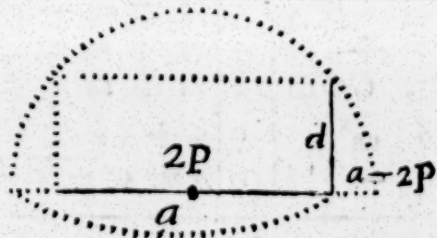
The Difference between the Base and Cathetus of any Right-angled Triangle, and the Perpendicular let fall from the Right angle upon the Hypothenuse, being given; thence to find the Hypothenuse, &c.

Let	1	$d = 15$	The Difference of the Sides
And	2	$p = 36$	
Quere a	3	$a =$	The Hypothenuse.
By Fig.	4	$d + e : p :: a : e$	
4	5	$de + ee = pa$	
Again,	6	$dd + 2de + 2ee = aa$.	By Theorem 11.



5 $\times$ 2	7	$dde + 2ee = 2pa$	
6 $-$ 7	8	$dd = aa - 2pa$.	Case 2.
8 $C \square$	9	$aa - 2pa + pp = dd + pp = 1521$.	
9 w^2	10	$a - p = \sqrt{dd + pp} = 39$	
10 $+ p$	11	$a = p + \sqrt{dd + pp} = 75$,	&c. for e . per Step 5.

The Geometrical Construction of this Case 2, viz. $aa - 2pa = dd$ may be performed in the very same Manner as the last Case was; that is, by making a Right-angled Parallelogram of the Co-efficient $2p$ and the $\sqrt{dd}$, viz. d , &c. As in the annexed Figure.



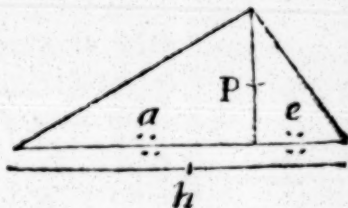
Then

Then will the *greater* Part of the Diameter to one End of the *Parallelogram* be $= a$, and the *lesser* Part will be $a - 2p$. For $a : d :: d : a - 2p$ by *Theorem 13*. Consequently, $aa - 2pa = dd$. Which was to be done.

P R O B L E M VIII.

The Hypothenufe of any Right-angled Triangle, and the Perpendicular let fall from the Right-angle upon the Hypothenufe, being given, to find the greater Segment of the Hypothenufe, &c.

Let	1	$b = 75$	The Hypothenufe
And	2	$p = 36$	
Then	3	$a + e = b$	Quere a
per Fig.	4	$a : p :: p : e$	
4	5	$\frac{pp}{a} = e$	
3 — a	6	$b - a = e$	
5, 6	7	$b - a = \frac{pp}{a}$	
7 $\times a$	8	$ba - aa = pp$	Case 3.
8 $+$	9	$aa - ba = -pp$	
9 $\bar{C}$ $\square$	10	$aa - ba + \frac{1}{4}bb = \frac{1}{4}bb - pp = 110, 25$	
10 w^2	11	$a - \frac{1}{2}b = \sqrt{\frac{1}{4}bb - pp} = 10, 5$	
11 $+$ $\frac{1}{2}b$	12	$a = \frac{1}{2}b \pm \sqrt{\frac{1}{4}bb - pp} = 48.$	Or, $a = 27.$



The Geometrical Construction of *Case 3*, viz. $ba - aa = pp$, may be thus performed: Draw a *Right-line* (of any convenient Length at Pleasure) and near its Middle erect a Perpendicular $= p$, viz. of the same Length with the *Root* of the *Resolvend*. From the top Point or upper End of that Perpendicular, set off Half the

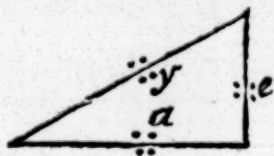


Length of the Co-efficient, viz. $\frac{b}{2}$ and upon the Point where $\frac{b}{2}$ just touches the first Line (with the same Distance) describe a Semicircle; then will its Diameter b be cut by the Perpendicular p into two Segments, which are the two Values of the Root a , viz. the *greater* and *lesser* Roots, both taken together, being always equal to the Co-efficient: (*vide* Page 201.) For $b - a : p :: p : a$ by *Theorem 13*. Ergo, $ba - aa = pp$. Which was to be done.

PROBLEM IX.

The Perimeter, i. e. the Sum of all the three Sides of any Right-angled Triangle, and its Area, being given, thence to find each Side.

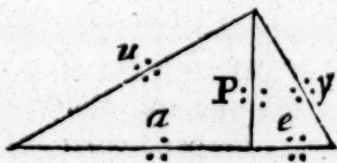
Viz. Let	1	$a + e + y = s = 234$	The Sum of the Sides.
And	2	$\frac{1}{2}ae = A$	The Area = 2340
Again,	3	$aa + ee = yy$	By Figure
2 $\times$ 4	4	$2ae = 4A$	
3 $+$ 4	5	$aa + 2ae + ee = yy + 4A$	
1 $-$ y	6	$a + e = s - y$	
6 $\odot^2$	7	$aa + 2ae + ee = ss - 2sy + yy$	
5, 7	8	$yy + 4A = ss - 2sy + yy$	
8 $\pm$	9	$2sy = ss - 4A = 45396$	
9 $\div$ 2s	10	$y = \frac{ss - 4A}{2s} = s - \frac{2A}{s} = 97$	The Hypothenufe.
6, 10	11	$a + e = s - y = 137$	
3 $-$ 4	12	$aa - 2ae + ee = yy - 4A = 49$	
12 $\sqrt{}$	13	$a - e = \sqrt{49} = 7$	
11 $+$ 13	14	$2a = 137 + 7 = 144$	
13 $\div$ 2	15	$a = 72$	The Base.
11 $-$ 15	16	$e = 137 - 72 = 65$	The Cathetus.



PROBLEM X.

In any Right-angled Triangle a Perpendicular being let fall from the Right-angle upon the Hypothenufe; if the Sum of each Segment, when added to its adjacent or next Side, be given, thence to find each Side, and the Segments.

Viz. If	1	$a + u = s = 108$	
And	2	$e + y = z = 72$	
To find		$a, e, u, y, \text{ and } p$	
1 $-$ a	3	$u = s - a$	
3 $\odot^2$	4	$uu = ss - 2sa + aa$	
4 $-$ aa	5	$uu - aa = ss - 2sa = pp$	
2 $-$ e	6	$z - e = y$	
6 $\odot^2$	7	$zz - 2ze + ee = yy$	
7 $-$ ee	8	$zz - 2ze = yy - ee = pp$	
5, 8	9	$zz - 2ze = ss - 2sa$	
By Fig.	10	$a : p :: p : e$	
10 $\therefore$	11	$ae = pp$	
5, 11	12	$ae = ss - 2sa$	

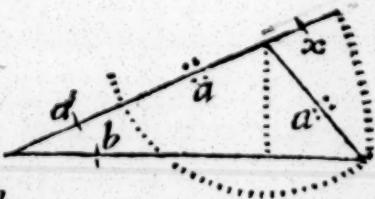


12	$\div a$	13	$e = \frac{ss - 2sa}{a}$	
13	$\times 2z$	14	$2ze = \frac{2zss - 4zsa}{a}$	
9	$+$	14	15	$zz = ss - 2sa + \frac{2zss + 4zsa}{a}$
15	$\times a$	16	$zza = ssa - 2saa + 2zss - 4zsa$	
16	$\pm$	17	$2saa + zza + 4zsa - ssa = 2zss$	
17	$\div 2s$	18	$aa + \frac{zza}{2s} + 2z - \frac{1}{2}sa = zs$	
Substitute		19	$2x = \frac{zz}{2s} + 2z - \frac{1}{2}s = 114$	
Then		20	$aa + 2xa = zs = 7776$	
20 C	$\square$	21	$aa + 2xa + xx = zs + xx = 11025$	
21	w^2	22	$a + x = \sqrt{zs + xx} = 105$	
22	$- x$	23	$a = \sqrt{zs + xx} - x = 48$	
1	$-$	23	$u = 60 = \text{The Base.}$	
per 13		25	$e = \frac{ss}{a} - 2s = 27$	
2	$-$	25	$y = 45 = \text{the Cathetus.}$	
23	$+$	25	27	$a + e = 75 = \text{the Hypothenufe.}$

PROBLEM XI.

The Difference of the Sides of any Oblique-angled plain Triangle, the Difference of the Segments of the Base, and the Difference between the greater Side and the Base, being given, to find the Base, &c.

Let	{	1	$d = \text{the Difference of the Sides} = 405$	
		2	$b = \text{the Difference of the Segments} = 495$	
		3	$x = 165 \text{ the Differ. of the greater Side and Base}$	
And		4	$a = \text{the least Side}$	
Then		5	$d + a + x = \text{the Base}$	
And		6	$d + a + x : d + 2a :: d : b$	
			By Theorem 16.	
6	$\therefore$	7	$db + ba + bx = dd + 2da$	
7	$\pm$	8	$2da - ba = db + bx - dd$	
8	$\div 2d - b$	9	$a = \frac{db + bx - dd}{2d - b} = \frac{118125}{315} = 375$	
1	$+$	9	10	$d + a = 780 = \text{the greatest Side.}$
3	$+$	10	11	$d + a + x = 945 = \text{the Base.}$



PROBLEM XII.

The Difference of the Sides of any plain Triangle, the Difference of the Segments of the Base, and the Perpendicular let fall from the Vertical Angle, being given, thence to find all the Sides.

Let	1	$d = 405$	} as before
	2	$b = 495$	
And	3	$p = 300$	
Quere	4	$a =$ the lesser Segment.	
Then	5	$b + 2a : d + 2e :: d : b.$	
5	6	$bb + 2ba = dd + 2de$	
6	7	$bb - dd + 2ba = 2de$	
Substitute	8	$2x = bb - dd = 81000$	
7,	8	$2x + 2ba = 2de$	
9	9	$x + ba = de$	
9	10	$\frac{x + ba}{d} = e$	
But	11	$pp + aa = ee$ By Theorem II.	
10 $\odot^2$	12	$\frac{xx + 2xba + bbaa}{dd} = ee$	
11,	12	$\frac{xx + 2xba + bbaa}{dd} = pp + aa$	
13	13	$xx + 2xba + bbaa = ppdd + ddaa$	
13	14	$xx + 2xba + bbaa = ppdd + ddaa$	
14	15	$bbaa - ddaa + 2xba = ppdd - xx$	
8,	15	$2xaa + 2xba = ppdd - xx$	
16	16	$aa + ba = \frac{ppdd}{2x} - \frac{1}{2}x$	
17	17	$aa + ba + \frac{1}{4}bb = \frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x$	
18	18	$a + \frac{1}{2}b + \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x}$	
19	19	$a = \sqrt{\frac{1}{4}bb + \frac{ppdd}{2x} - \frac{1}{2}x} - \frac{1}{2}b = 225$	
20	20	$2a = 450$	
2	21	$b + 2a = 945$ the Base.	
10, Num.	22	$e = 375$ the lesser Side.	
1	23	$d + e = 780$ the greater Side.	

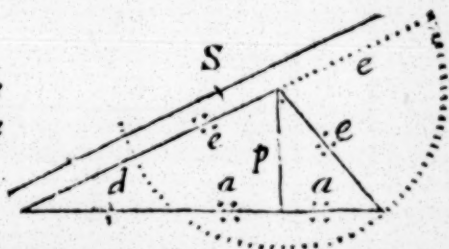


PROBLEM XIII.

The Sum of the two Sides of any plain Triangle, the Difference of the Segments of the Base, and the Perpendicular let fall from the Vertical

Vertical Angle upon the Base, being given, thence to find the Base and the Sides.

Let	{	1	$s = 1155$ the Sum of the Sides.
		2	$d = 495$ the Difference of the Segments.
		3	$p = 300$ the Perpendicular.
Put	{	4	$a =$ the least Segment.
		5	$e =$ the least Side.
Then		6	$d + 2a =$ the Base.
And		7	$s - 2e =$ the Difference of the Sides.
Per Fig.	{	8	$d + 2a : s :: s - 2e : d$
		9	$aa + pp = ee$
9 u^2		10	$\sqrt{aa + pp} = e$
8 $\cdot$		11	$dd + 2da = ss - 2se$
11 $+$		12	$2se = ss - dd - 2da$
Suppose		13	$2x = ss - dd$
Then		14	$2se = 2x - 2da$
14 $\div 2s$		15	$e = \frac{x - da}{s}$
10,	15	16	$\frac{x - da}{s} = \sqrt{aa + pp}$
16 $\odot^2$		17	$\frac{xx - 2xda + ddaa}{ss} = aa + pp$
17 $\times ss$		18	$xx - 2xda + ddaa = ssaa + sspp$
18 $+$		19	$ssaa - ddaa + 2xda = xx - sspp$
13	19	20	$2xaa + 2xda = xx - sspp$
20 $\div 2x$		21	$aa + da = \frac{1}{2}x - \frac{sspp}{2x}$, &c. as before.
21, hence		22	$a = 225$
22 $\times 2$		23	$2a = 450$
2 $+$	23	24	$d + 2a = 945$ the Base.
10, Num.		25	$e = 375$ the lesser Side.
1 $- 25$	26		$s - e = 780$ the greater Side.



PROBLEM XIV.

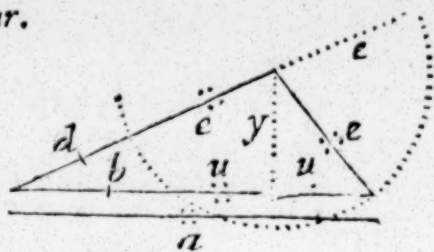
The Area of any Oblique-angled plain Triangle, the Difference of the Sides, and the Difference of the Segments of the Base, being given, thence to find the Base, &c.

Let	{	1	$A = 141750 =$ the Area.
		2	$d = 405$
		3	$b = 495$

U u

Put

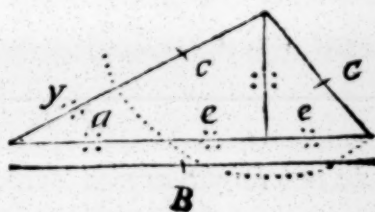
Put {	4	$y =$ the Perpendicular.
Then	5	$a =$ the Base.
	6	$\frac{1}{2} ya = A$
Per Fig.	7	$a : d + 2e :: d : b$
7	8	$ba = dd + 2de$
8 — dd	9	$ba - dd = 2de$
9 $\odot^2$	10	$bbaa - 2ddba + dddd = 4ddee$
Per Fig.	11	$\frac{a-b}{2} = u$ the lesser Segment of the Base.
11 $\odot^2$	12	$\frac{aa - 2ba + bb}{4} = uu$
6 $\times$ 2	13	$ya = 2A$
13 $\div a$	14	$y = \frac{2A}{a}$
14 $\odot^2$	15	$yy = \frac{4AA}{aa}$
Per Fig.	16	$yy + uu = ee = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
10 $\div 4dd$	17	$\frac{bbaa - 2ddba + dddd}{4dd} = ee$
16, 17	18	$\frac{bbaa - 2ddba + d^4}{4dd} = \frac{4AA}{aa} + \frac{aa - 2ba + bb}{4}$
18 $\times aa$	19	$\frac{bba^4 - 2ddba^3 + d^4aa}{4dd} = 4AA + \frac{a^4 - 2ba^3 + bba^3}{4}$
19 $\times 4dd$	20	$\begin{cases} bba^4 - 2ddba^3 + d^4aa = 16AAdd + dda^4 \\ - 2ddba^3 + ddbba^2 \end{cases}$
20 $+$	21	$bba^4 - dda^4 + d^4a^2 - ddbba^2 = 16AAdd$
21 $\div$	22	$aaaa - ddaa = \frac{16AAdd}{bb - dd}$
22 C $\square$	23	$aaaa - ddaa + \frac{1}{4} dddd = \frac{16AADD}{bb - dd} + \frac{1}{4} d^4$
23 w^2	24	$aa - \frac{1}{2} dd = \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4} d^4}$
24 $+$ $\frac{1}{2} dd$	25	$aa = \frac{1}{2} dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4} d^4}$
25 w^2	26	$a = \sqrt{\frac{1}{2} dd + \sqrt{\frac{16AAdd}{bb - dd} + \frac{1}{4} d^4}} = 945$



PROBLEM XV.

There is an Oblique-angled plain Triangle, wherein a Perpendicular is let fall from the Vertical Angle upon the Base; the least Side and the Base are given; and the Rectangle of the Difference of the Sides into the least Side is equal to the Square of the Difference of the Segments of the Base: 'Tis requir'd to find the Segments of the Base, &c.

Let	{	1	$c = 56 =$ the least Side.
		2	$B = 92 =$ the Base.
And		3	$a + 2e = B$
Put		4	$y =$ the Difference of the Sides.
Then		5	$cy = aa$ by the Question.
By Figure		6	$B : 2c + y :: y : a$, for $B = a + 2c$
6	$\therefore$	7	$Ba = 2cy + yy$
5	$\times$	8	$2cy = aa$
7	$-$	9	$Ba - 2aa = yy$
5	$\odot^2$	10	$ccyy = aaaa$
10	$\div$	11	$yy = \frac{aaaa}{cc}$
9,		12	$Ba - 2aa = \frac{aaaa}{cc}$
12	$\times$	13	$cc Ba - 2ccaa = aaaa$
13	$\div$	14	$cc B - 2cca = aaa$
14	$+ 2cca$	15	$aaa + 2cca = ccB$
15 in Num.		16	$aaa + 6272a = 288512$



The Value of a , in this Equation, may be found as in the Examples Page 238, viz. by putting $r + e = a$, &c. as in those Examples you will find $a = 37,55502$, &c.

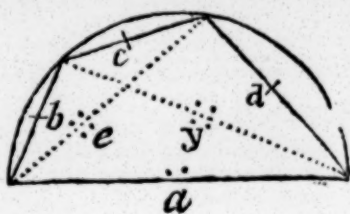
PROBLEM XVI.

The three Chords or Subtenses of three Arches compleating a Semicircle being each given, thence to find the Diameter of that Circle. That is,

Any Trapezium being inscrib'd in a Semicircle, if one of its Sides be the Diameter, and the other three Sides be given, thence to find the Diameter or fourth Side.

Let $\left\{ \begin{array}{l} 1 \ b = 3 \\ 2 \ c = 4 \\ 3 \ d = 5 \end{array} \right\}$ the 3 Sides.
 Quære 4 $a =$ the Diam. sought

Draw the two Diagonals
 e and y



Then 5 $ca + bd = ey$. By Theorem 19.

And $\left\{ \begin{array}{l} 6 \ aa - bb = yy \\ 7 \ aa - dd = ee \end{array} \right\}$ By Theorem 10 and 17.

5	$\odot^2$	8	$ccaa + 2bdca + bbdd = eeyy$
6	$\times$	9	$aaaa - bbaa - ddaa + bbdd = eeyy$
8,	$=$	10	$aaaa - bbaa - ddaa = ccaa + 2bdca$
10	$- a$	11	$aaa - bba - dda - cca = 2bdc$
11	$- cca$	12	$aaa - bba - dda - cca = 2bdc$
12, Numb.		13	$aaa - 50a = 120$

This Equation being solv'd as in Example 2, Page 240, you will find $a = 8,05581$, &c.

PROBLEM XVII.

In any Right-angled Triangle, the Area and the Sum of the Hypotenuse, when added to either Side, being given, thence to find the Sides, &c.

Suppose $\left\{ \begin{array}{l} 1 \ \frac{ae}{2} = A = 1350 \text{ the Area.} \\ 2 \ y + e = s = 120 \text{ the Sum, \&c.} \\ 3 \ \text{Quære } a, e, \text{ and } y \end{array} \right.$

1 $\times$ 2 4 $ae = 2A$

4 $\div a$ 5 $e = \frac{2A}{a}$

Per Fig. 6 $aa + ee = yy$

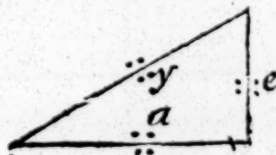
2 $- e$ 7 $y = s - e$

5, 7 8 $y = s - \frac{2A}{a}$

8 $\odot^2$ 9 $yy = ss - \frac{4sA}{a} + \frac{4AA}{aa}$

5 $\odot^2$ 10 $ee = \frac{4AA}{aa}$

10 $+ aa$ 11 $aa + ee = \frac{4AA}{aa} + aa$



6, 9, 11

$$\begin{array}{ll}
 6, 9, 11 & 12 \quad \frac{4AA}{aa} + aa = yy = ss - \frac{4sA}{a} + \frac{4AA}{aa} \\
 12, \text{That is} & 13 \quad aa = ss - \frac{4sA}{a} \\
 13 \times a & 14 \quad aaa = ssa - 4sA \\
 14 + & 15 \quad ssa - aaa = 4sA \\
 15, \text{in Num.} & 16 \quad 14400a - aaa = 648000
 \end{array}$$

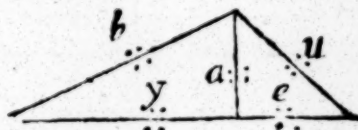
The Value of a , in this Equation, may be found as in the third Example, Page 241; that is, by making $r + e = a$, &c. it will be found that $a = 60$.

PROBLEM XVIII.

There is an Oblique-angled plain Triangle, wherein a Perpendicular is let fall from the Vertical Angle upon the Base; the Sum of each Segment of the Base, when added to its adjacent or next Side, and the Area of the Triangle, are given, to find the Perpendicular and each Side.

$$\begin{array}{ll}
 \text{Let } \left\{ \begin{array}{l} 1 \quad y + b = z = 1500 \\ 2 \quad e + u = s = 600 \end{array} \right\} & \text{Quære } y, b, e, \text{ and } u \\
 \text{And} & 3 \quad A = \text{the Area} = 141750 \\
 \text{Then} & 4 \quad a = \text{the Perpendicular sought.} \\
 & 5 \quad y + e \times \frac{1}{2} a = A
 \end{array}$$

$$\begin{array}{ll}
 5 \times 2 \div a & 6 \quad y + e = \frac{2A}{a} \\
 \text{Per Fig. } \left\{ \begin{array}{l} 7 \quad yy + aa = bb \\ 8 \quad ee + aa = uu \end{array} \right. & \\
 1 - y & 9 \quad b = z - y \\
 2 - e & 10 \quad u = s - e \\
 9 \odot^2 & 11 \quad bb = zz - 2zy + yy \\
 10 \odot^2 & 12 \quad uu = ss - 2se - ee \\
 7, & 11 \quad 13 \quad zz - 2zy = aa \\
 8, & 12 \quad 14 \quad ss - 2se = aa \\
 13 \quad + & 15 \quad zz - aa = 2zy \\
 14 \quad + & 16 \quad ss - aa = 2se
 \end{array}$$



$$\begin{array}{ll}
 15 \div 2z & 17 \quad \frac{zz - aa}{2z} = y \\
 16 \div 2s & 18 \quad \frac{ss - aa}{2s} = e \\
 17 \div 18 & 19 \quad \frac{zz - aa}{2z} + \frac{ss - aa}{2s} = y + e
 \end{array}$$

Having found the Value of a from the 24th Step, e and y will be easily found by these two Steps, and b, u , by the 9th and 10th Step.

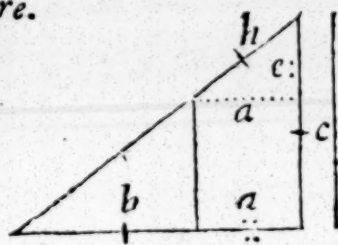
6,	19	20	$\frac{zz - aa}{2z} + \frac{ss - aa}{2s} = \frac{2A}{a}$
20	×	22	$zz - aa + \frac{zzs - zaa}{s} = \frac{4za}{a}$
21	×	5	$zzs - saa + zss - zaa = \frac{4zA}{a}$
22	×	a	$zzsa - saaa + zssa - zaaa = 4zAs$
23, Numb.		24	$900000a - aaa = 243000000$

Here $a = 300$ found as in the last Problem.

PROBLEM XIX.

There is a Right-angled Triangle, wherein a Right-line is drawn parallel to the Cathetus; there is given the Cathetus, that Segment of the Hypothenufe next to the Cathetus, and the alternate Segment of the Base; thence to find the Base, &c.

viz. Let	1	$b = 20 . c = 24 . \text{ and } h = 15$
Then	2	$b + a = \text{the Base. Quære } a$
Here	3	$b + a : c :: a : e \text{ per Figure.}$
And	4	$aa + ee = hb \text{ per Figure.}$
3	∴	5 $\frac{ca}{b + a} = e$
5	⊙ ²	6 $\frac{ccaa}{bb + 2ba + aa} = ee$
4	—	7 $hb - aa = ee$
6,	7	8 $\frac{ccaa}{bb + 2ba + aa} = hb - aa$
8	×	9 $ccaa = hbbb - bbaa + 2hbba - 2ba^3 + hbba - a^4$
9	+	10 $a^4 + 2baaa + ccaa + bbqa - hbba - 2hbba = hbbb$
That is,	11	$aaaa + 40aaa + 751aa - 9000a = 90000$



For a Solution of this Equation, let it be made

$$aaaa + baaa + caa - da = G \quad \text{Viz. } \begin{cases} b = 40 & c = 75 \\ d = 9000 & G = 90000 \end{cases}$$

Put $r + e = a$

$$\text{Then } \begin{cases} r + 4rrre + 6rree = a^4 \\ brrr + 3brre + 3bree = baaa \\ cr + 2cre + cee = caa \\ -dr - de = -da \end{cases} = G = 90000$$

Let $r = 10$

Then

$$\text{Then } \left\{ \begin{array}{l} + 10000 + 4000e + 600ee \\ + 40000 + 12000e + 1200ee \\ + 75100 + 15020e + 751ee \\ - 90000 - 9000e \end{array} \right\} = G = 90000$$

That is, $35100 + 22020e + 2551ee = 90000$

Hence it will be $22020e + 2551ee = 54900$

Consequently, $8,63e + ee = 21,52 = D$

And $\frac{D}{8,63 + e} = e$

Operation, $8,63 \overline{) 21,52} \quad (2,1 = e$
 $+ e = 2,1 \quad 20$

1. Divisor $= 10 \quad 1,52 \quad \text{First } r = 10$
 2. Divisor $= 10,7 \quad 1,07 \quad + e = 2,1$

45 &c. $r + e = 12,1 = r$ for a second

Operation, which being involv'd, and multiply'd into the *Co-efficients*, as before, will produce these *Numbers*:

$$\left\{ \begin{array}{l} + 21435,8881 + 7086,24e + 878,46ee \\ + 70862,4400 + 17569,20e + 1452,00ee \\ + 109953,9100 + 18174,20e + 751,00ee \\ - 108900,0000 - 9000,00e \end{array} \right\} = C.$$

Viz. $93352,2381 - 33829,64e + 3081,46ee = 90000$

Here, because $93352,2381 > 90000$ therefore $12,1 > a$, and therefore it must be made $r - e = a$, which will produce the same *Numbers*, only all the second Signs must be changed.

Thus, $93352,2381 - 33829,64e + 3081,46ee = 90000$ from whence will arise this *Æquation*:

$+ 33829,64e - 3081,46ee = 3352,2381$

Consequently, $10,9784e - ee = 1,08787332 = D$

Operation, $10,9784 \overline{) 1,08787332} \quad (0,0999 = e$
 $- e = 0,0999 \quad 9792$

1. Divisor $10,88 \quad 108673 \quad \text{Last } r = 12,1$
 2. Divisor $10,879 \quad 97911 \quad - e = 0,0999$
 3. Divisor $10,8785 \quad 1076232 \quad r - e = 12,0001 = e$
 979065

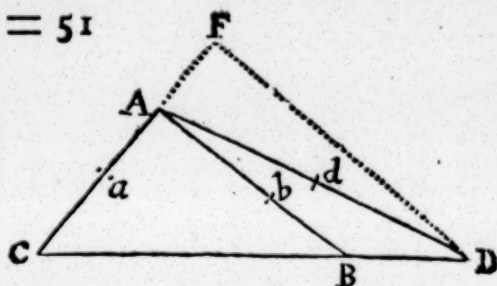
&c.

PROBLEM XX.

In the Oblique-angled Triangle *CAD*, there is given the Side *AD*, and the Sum of the Sides *AC* + *CD*; also within the Triangle is given the

the Line AB perpendicular to the Side CA ; thence to find the Side CA , &c.

Let $\left\{ \begin{array}{l} 1 \text{ } CA + CD = s = 51 \\ 2 \text{ } AD = d = 32 \\ 3 \text{ } AB = b = 21 \\ \text{And } 4 \text{ } CA = a \text{ sought.} \\ \text{Then } 5 \text{ } s - a = CD \end{array} \right.$



Suppose the Line DF

parallel to AB ; CA being produced to F
 $\triangle CAB$, and $\triangle CFD$ will be alike.

Then And 6 $BC : CA :: DC : CF$

But 7 $BC = \sqrt{bb + aa}$. Let $AF = e$, and $FD = y$

6, 7 8 $\sqrt{bb + aa} : a :: s - a : a + e$

8 9 $\frac{sa - aa}{\sqrt{bb + aa}} = a + e$

5 $\odot^2$ 10 $ss - 2sa + aa = \square CD$

Per Fig. 11 $ss - 2sa + aa = aa + 2ae + ee + yy = \square CF + \square FD$

11 $- aa$ 12 $ss - 2sa = 2ae + ee + yy$

But 13 $dd = ee + yy = \square AF + \square FD$

12 $- 13$ 14 $ss - 2sa - dd = 2ae$

Let 15 $2x = ss - dd$

14, 15 16 $x - sa = ae$

16 $\div a$ 17 $\frac{x - sa}{a} = e$

17 $+ a$ 18 $\frac{x - sa + aa}{a} = a + e$

9 $\odot^2$ 19 $\frac{ssaa - 2saaa + a^4}{bb + aa} = \square a + e$

18 $\odot^2$ 20 $\frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} = \square a + e$

19, 20 21 $\left\{ \begin{array}{l} \frac{ssaa - 2saaa + a^4}{bb + aa} \\ = \frac{xx - 2xsa + 2xaa + ssaa - 2sa^3 + a^4}{aa} \end{array} \right.$

This Equation being brought out of the Fractions, and into Numbers, will become $-2018a^4 + 125409a^3 - 2464230,25a^2 + 35468307a = 274183922,25$; which being divided by 2018, the Co-efficient of the highest Power of a , will be $-a^4 + 62,1456a^3 - 1221,125a^2 + 17575,9697a = 135869,138875$, &c.

And from hence the *Value* of *a* may be found, as in the *last Problem*, due Regard being had to the *Signs* of every *Term*.

This Work of *reducing*, or preparing *Equations* for a *Solution* by *Division*, hath always been taught both by *ancient* and *modern* Writers of *Algebra*, as a Work so necessary to be done, that they do not so much as give a Hint at the *Solution* of any *adfectèd Equation* without it.

Now it very often happens, that, in *dividing* all the *Terms* of an *Equation*, some of their *Quotients* will not only run into a long *Series*, but also into imperfect *Fractions* (as in this *Equation* above) which renders the *Solution* both tedious and *imperfect*.

To remedy that *Imperfection*, I shall here shew how this *Equation* (and consequently any other) may be *resolv'd* without such *Division* or *Reduction*.

Let $b = 2018$. $c = 125409$. $d = 2464230,25$

$f = 35468307$. And $G = 274183922,25$

Then the *precedent Equation* will stand thus:

$$-baaaa + caaa - daa + fa = G$$

Put $r + e = a$ as before.

$$\text{Then will } \left\{ \begin{array}{l} -br^4 - 4brrre - 6brree = -ba^4 \\ + cr^3 + 3crrr + 3crré = +ca^3 \\ - drr - 2dré - dèè = -daa \\ + fr + fè \dots \dots = +fa \end{array} \right\} = G$$

This is plain and easily conceived. The next Thing will be, how to estimate the first *Value* of r ; and, to perform that, let G be *divided* by b , only so far as to determine how many *Places* of whole *Numbers* there will be in the *Quotient*; consequently, how many *Points* there must be (according to the *Height* of the *Equation*.)

$$\begin{array}{r} \text{Thus } b = 2018) G = 274183922,25 \text{ (130000} \\ \quad \quad \quad \underline{2018} \\ \quad \quad \quad 7238, \text{ \&c.} \end{array}$$

Now from hence one may as easily guess at the *Value* of r , as if all the *Terms* had been *divided*. That is, I suppose $r = 10$, which being *involved*, &c. as the *Letters above direct*, will be

X x

— 20880000

$$\begin{array}{r}
 - 20880000 - 8072000e - 1210800ee \\
 + 125409000 + 37622700e + 3762270ee \\
 - 246423025 - 49284605e - 2464230, 25ee \\
 + 354683070 + 35468307e
 \end{array} \} = G$$

Viz. $213489045 + 15734402e + 87239,75ee = 2741839 \&c.$

Hence $15734402e + 87239,75ee = 60694877,25$

Consequently, $180,3e + ee = 695,72 = D$

$$\text{And } \frac{D}{180,3 + e} = e$$

Operation. $180,3) \quad 695,72 \quad (3,7 = e$

$$\begin{array}{r}
 + e = 3,7) \quad 549 \\
 1. \text{ Divisor} = 183 \quad 146,72 \quad \text{First } r = 10 \\
 2. \text{ Divisor} = 184,0 \quad 128,80 \quad + e = 3,7
 \end{array}$$

&c.

$r + e = 13,7 = r$ for a

second Operation, with which you may proceed, as in the last Problem, and so on to a third Operation, if Occasion require such Exactness. But this may be sufficient to shew the Method of resolving any *affected Equation*, without reducing it; which is not only very exact, but also very ready in Practice, as will fully appear in the last Chapter of this Part, concerning the *Periphery* and *Area* of the *Circle*, &c. wherein you will find a farther Improvement in the *Numerical Solution* of *High Equations* than hath hitherto been publish'd.

CHAP. V.

Practical Problems, and Rules for finding the Superficial Contents, or Area's of Right-lin'd Figures.

BEfore I proceed to the following Problems, it may be convenient to acquaint the Learner, that the *Superficies* or *Area* of any *Figure*, whether it be *Right-lin'd* or *Circular*, is compos'd or made up of *Squares*, either greater or less, according to the different *Measures* by which the *Dimensions* of the *Figures* are taken or measur'd.

That is, if the *Dimensions* are taken in *Inches*, the *Area* will be compos'd of *square Inches*; if the *Dimensions* are taken in *Feet*, the *Area* will be compos'd of *square Feet*; if in *Yards*, the *Area* will be *square Yards*; and if the *Dimensions* are taken by *Poles* or *Perches* (as in surveying of Land, &c.) then the *Area* will be *square Perches*,

&c.

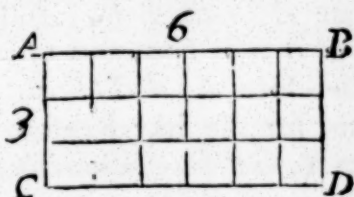
&c. These Things being understood, and the *Definitions* in the 283d and 284th *Pages* well consider'd, will help to render the following *Rules* very easy.

P R O B L E M I.

To find the Superficial Content, or Area of a Square; or of any Right-angled Parallelogram.

RULE. { Multiply the Length into its Breadth, and the Product will be the Area requir'd. (See Lemma 1. Page 302.)

Example. Suppose the Line $AB = 6$ Yards, and the Breadth AC or $BD = 3$ Yards, then $AB \times AC = 6 \times 3 = 18$ will be the Number of square Yards contain'd in the Area of the Parallelogram $ABCD$. This is so evident by the Figure only, that it needs no Demonstration.

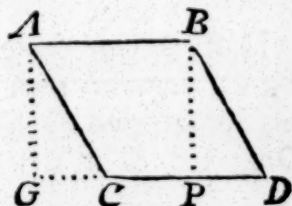


P R O B L E M II.

To find the Area of any Oblique-Triangled Parallelogram, viz. either of a Rhombus or Rhomboides.

RULE. { Multiply the Length into its perpendicular Height (or Breadth) and the Product will be the Area requir'd.

That is, the Side $AB \times BP =$ the Area of the Rhombus $ABCD$. For if BP be drawn perpendicular to CD , and AG be made parallel to BP , then will $GC = PD$ and $GP = CD$. Consequently $\triangle AGC = \triangle BPD$, and $\square ABGP =$ Rhombus $ABCD$. But $AB \times BP = \square ABGP$. Therefore $AB \times BP$, or $CD \times BP =$ the Area of the Rhombus $ABCD$.



Example. Suppose the Side $AB = 23$ Inches, and the Perpendicular $BP = 17,5$ Inches (being the shortest or nearest Distance between the two Sides, AB and CD .) then $AB \times BP = 23 \times 17,5 = 402,5$ square Inches, being the Area of the Rhombus required.

The like may be done for any Rhomboides whose Length and perpendicular Breadth is given.

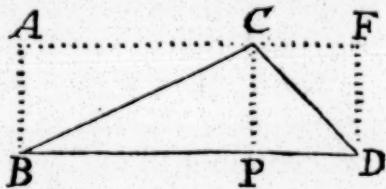
PROBLEM III.

To find the Superficial Content, or Area of any plain Triangle.

Every plain Triangle is equal to half its circumscribing Parallelogram, (41. e. 1.) which affords the following Rule :

RULE. { Multiply the Base of the given Triangle into half its perpendicular Height, or half the Base into the whole Perpendicular, and the Product will be the Area.

That is, $BD \times \frac{1}{2} CP$, or $\frac{1}{2} BD \times CP = \text{Area of } \triangle BCD$. For $AC = BP$, $AB = CP$, and BC is common to both $\triangle ABC$ and $\triangle BCP$; therefore $\triangle ABC = \triangle BCP$, and for the like Reasons $\triangle CFD = \triangle CPD$. Therefore $\triangle BCP + \triangle CPD = \frac{1}{2} \square ABCD$. Consequently $\frac{1}{2} BD \times CP$, or $BD \times \frac{1}{2} CP$ will be the Area of $\triangle BCD$.



Example. Suppose the Base $BD = 32$ Inches, and the perpendicular Height $CP = 14$ Inches.

Then $\frac{1}{2} BD \times CP = 16 \times 14 = 224$. Or $BD \times \frac{1}{2} CP = 32 \times 7 = 224$. Or thus, $32 \times 14 = 448$. Then $2) 448$ (224 = the Area of the Triangle BCD in square Inches.

PROBLEM IV.

To find the Superficies, or Area of any Trapezium.

First, divide the given Trapezium into two Triangles, by drawing a Diagonal from one of its acute Angles to the opposite Angle; and let fall two Perpendiculars (from the other two Angles) upon the Diagonal, as in the following Figure. Then

RULE. { Multiply half the Diagonal into the Sum of the two Perpendiculars, or half the Sum of the Perpendiculars into the Diagonal, and the Product will be the Area.

That is, $\frac{1}{2} AC \times \overline{BP + ED}$. Or $AC \times \frac{1}{2} \overline{BP + ED} = \text{Area of the Trapezium } ABCD$.

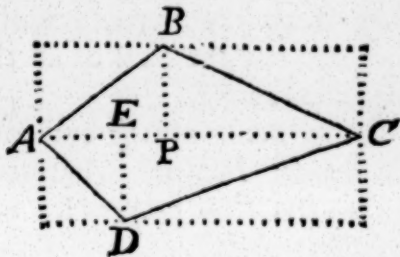
For the $\triangle ABC$ is Half its circumscribing Parallelogram; and the $\triangle ACD$ is also Half of its circumscribing Parallelogram, as hath been prov'd at the last Problem.

Consequently,

Consequently, $BP + ED \times \frac{1}{2} AC$, or $\frac{1}{2} BP + \frac{1}{2} ED \times AC$ will be the *Area* of the *Trapezium*, as above.

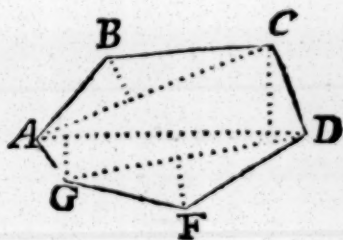
Example. Suppose the *Diagonal* $AC = 33$ Feet, and the *Perpendicular* $BP = 15$ Feet, and the *Perpendicular* $ED = 14$ Feet. Then $BP + ED = 29$ Feet, and $BP + ED \times \frac{1}{2} AC = 29 \times 16,5$

$= 478,5$. Or $AC \times \frac{1}{2} BP + \frac{1}{2} ED = 33 \times \frac{29}{2} = 478,5$. Or thus, $29 \times 33 = 957$. Then 2) 957 (478,5 any of these *Products* are the *Area* of the *Trapezium* $ABCD$.



PROBLEM V.

To find the *Superficial Content* or *Area* of any irregular Polygon, or many sided Figure, which by some Authors is call'd a *Triangulate*, because (as I suppose) it must be divided into *Triangles*, as in the annexed Figure $ABCD$ FG ; by which it is evident, that the *Sum* of the *Area's* of all those *Triangles*, found as in the last *Problem*, &c. will be the *Area* of their *circumscribing Polygon*.



PROBLEM VI.

To find the *Superficies*, or *Area* of any regular Polygon, viz. of any regular *Pentagon*, *Hexagon*, *Heptagon*, *Octagon*, &c.

General RULE. $\left\{ \begin{array}{l} \text{Multiply half the Sum of its Sides into the Radius} \\ \text{of the inscrib'd Circle, or half the said Radius into} \\ \text{the Sum of the Sides, and the Product will be the} \\ \text{Area required.} \end{array} \right.$

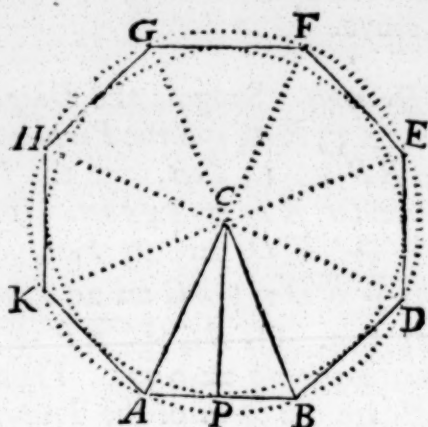
That is, $\frac{AB + BD + DE + EF + FG + GH + HK + KA}{2} : \times CP$

$=$ the *Area* of the annexed *Octagon*; wherein it is evident, that its *Area* is compos'd of so many equal *Isoceles Triangles* as there are *Numbers* of *Sides* in the *Polygon*, viz. of eight *Isoceles Triangles*, whose *Bases* are the *Sides* of the *Octagon*, viz. $AB = BD = DE$, &c. And the *Sides* of those *Triangles*, CA, CB, CD , &c. are the *Radius's* of the *circumscribing Circle*; and their *perpendicular Heights*, viz. PP , is the *Radius* of the *inscrib'd Circle*.

But the *Area* of any one of those *Triangles* is $\frac{1}{2} AB \times CP$ by *Problem 3*. Consequently the *Sum* of all their *Area's* will be CP into half the *Sum* of all their *Bases*, as above.

This, being equally evident in all regular *Polygons* whatsoever, makes the *Rule* general for finding their *Area's*.

Now, because it is requir'd to have the *Radius* of the propos'd *Polygon's* inscrib'd *Circle*, I shall here insert (and demonstrate) the *Proportions* that are between the *Sides* of several regular *Polygons* and the *Radius's* both of their inscrib'd and circumscribing *Circles*; the one will help to delineate or project the *Polygon* (if Occasion require it) and the other will help to find its *Area*.



And First, Of an Equilateral Triangle.

The Side of any *Equilateral plain Triangle* is in *Proportion* to the *Radius* of

$$\text{its } \left\{ \begin{array}{l} \text{Circumscribing Circle,} \\ \text{Inscrib'd Circle} \\ \text{Perpendicular Height,} \end{array} \right\} \text{As } 1 : \text{To } \left\{ \begin{array}{l} 0,57735027 \text{ \&c.} \\ 0,28867513 \text{ \&c.} \\ 0,86602540 \text{ \&c.} \end{array} \right.$$

$$\text{i. e. } \left\{ \begin{array}{l} AB : CD :: 1 : 0,57735027 \\ AB : CG :: 1 : 0,28867513 \\ AB : AG :: 1 : 0,86602540 \end{array} \right.$$

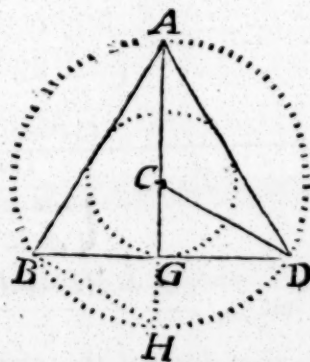
Demonstration.

Let $AB = BD = 1$, then will $BG = GD = 0,5$; but $\square AB - \square BG = \square AG$ by *Theorem 11*. That is, $1 - 0,25 = 0,75 = \square AG$, consequently, $\sqrt{0,75} = 0,86602540 = AG$:

Then $AG : AB :: AB : AH$, by *Theorem 13*, that is, $0,8660254 : 1 :: 1 : 1,15470054 \text{ \&c.} = AH$, then $\frac{1}{2} AH = 0,57735027 = AG$. Again, $AG : DG :: DG : CG$, that is, $0,8660254 : 0,5 :: 0,5 : 0,28867513 = CG$. Q. E. D.

Now, by the Help of the *First* of these *Proportions*, it will be easy to resolve the following *Problem*.

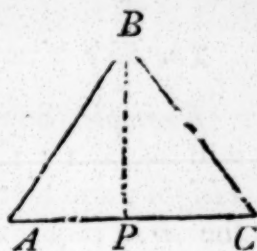
P R O-



PROBLEM VII.

The Side of any Equilateral plain Triangle being given, to find its Area.

Example, Suppose the Side of the propos'd Triangle ABC to be 25 Inches, viz. $AB = BC = CA = 25$
First $1 : 0,866254 :: AB = 25 : 21,650635$
 $= BP$ by *Theorem 13*. Then $AP (= \frac{1}{2} AC) \times BP =$ the Area of $\triangle ABC$ by *Rule*
to *Problem 3*, that is, $12,5 \times 21,650635 =$
 $270,6329$ the Area in square Inches.



Or this *Problem* may be otherwise resolv'd thus:
Let $b = AP = AC$. Then $2b = AB$. But
 $\square AB - \square AP = \square BP$. By *Theorem 11*. That is, $4bb -$
 $bb = 3bb = \square BP$. Consequently, $\sqrt{3bb} = BP$. Then b
 $\sqrt{3bb} = BP \times \frac{1}{2} AC$, viz. $\sqrt{3bb} \times \sqrt{3} =$ the Area of the
Triangle.

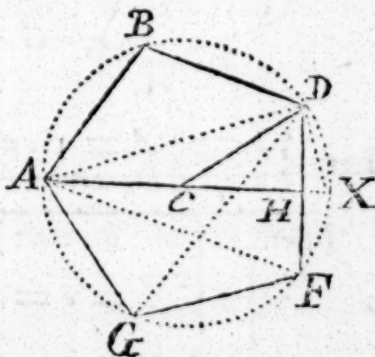
Secondly, For a Pentagon.

The Side of any Pentagon is in Proportion to the Radius of
its { *Circumscribing Circle,* } As 1 : To { 0,85065080 &c.
Inscrib'd Circle, } { 0,68819096 &c.
Perpendicular Height, } { 1,53884176 &c.

Viz. { $AB : AC :: 1 : 0,85065080$
 $AB : CH :: 1 : 0,68819096$
 $AB : AH :: 1 : 1,53884176$

Demonstration.

Let $AB = 1$. And draw the Diagonals AD , AF , and DG , which will be equal to one another. Then will $AG \times DF + AD \times GF = AF \times DG$ by *Theorem 19*. Consequently, $AG \times DF = AF \times DG : - AD \times GF$, that is, $\square AB = \square AD : - AD \times GF = 1$ (because $AB = AG = DF$, and $AD = AF = DG$) hence it will be $AD = 1,61803398$, then $\square AD - \square DH = \square AH$ by *Theor. 11*. But $DH = \frac{1}{2} AB$, therefore $\sqrt{\square AD - \frac{1}{4} \square AB} = AH = 1,53884176$. Again, $AH : AD :: AD : AX = 2 AC$. For $\triangle AHD$ and $\triangle ADX$ are alike.



Ergo

Ergo $\frac{\square AD}{AH} = 2 AC = 1,70130161$. Hence $AC = 0,85065080$

But $AH - AC = CH = 0,68819096$, &c. Q. E. D.

From hence it will be easy to resolve the following Problem.

PROBLEM VIII.

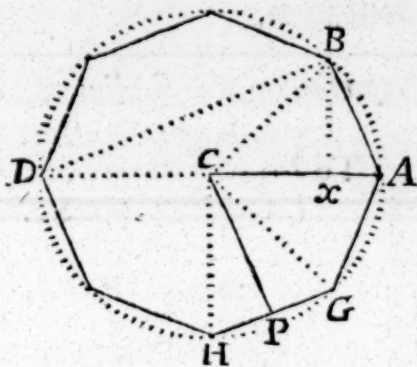
The Side of any regular Pentagon being given, to find its Area.

Example. Suppose the given Side to be 15 Inches long, then it will be, as $1 : 1,53884176 :: 15 : 22,0826264$ the perpendicular Height; and by the general Rule $22,0826264 \times \frac{1}{2} = 165,619698$ the Area requir'd.

Thirdly, For an Octagon.

The Side of any regular Octagon is in Proportion to the Radius of its
 { Circumscribing Circle, As 1 : to 1,30656296, &c.
 { Inscrib'd Circle, As 1 : to 1,20710678, &c.

Viz. $\begin{cases} BA : CA :: 1 : 1,30656296 \\ BA : CP :: 1 : 1,20710678 \end{cases}$



Demonstration.

Draw the Right Line DB, and from the Point B let fall the Perpendicular Bx upon the Diameter DA.

Then will $\triangle DBA$ and $\triangle Dx B$ be alike, by Theorem 10 and 12.

Let $\begin{cases} b = BA = 1. a = CA \\ e = BD, \text{ and } y = Bx \end{cases}$

Then $1 \quad 2a : b :: e : y. \text{ viz. } DA : BA :: DB : Bx$

$1 \quad \therefore 2 \quad \frac{2ay}{b} = e = DB$

$2 \quad \odot^2 \quad 3 \quad \frac{4aayy}{bb} = ee = \square DB$

But $4 \quad 4aa - \frac{4aayy}{bb} = bb$

That is $5 \quad \square DA - \square DB = \square BA. \text{ By Theorem 11.}$
 $4 \times bb \quad 4bbaa - 4aayy = bbbb$

Again $6 \quad \begin{cases} \frac{1}{2} aa = yy. \text{ For } Cx = Bx \\ \text{and } \square Cx + \square Bx = \square CB = aa \end{cases}$

5,	6	7	$4bbaa - 2a^4 = b^4$. Or $2a^4 - 4bbaa = -b^4$
7	$\div 2$	8	$aaaa - 2bbaa = -\frac{1}{2}b^4$
8	C □	9	$a^4 - 2bbaa + b^4 = b^4 - \frac{1}{2}b^4 = \frac{1}{2}b^4$
9	w^2	10	$aa - bb = \sqrt{\frac{1}{2}b^4}$
10	$+ bb$	11	$aa = bb + \sqrt{\frac{1}{2}b^4}$
11	w^2	12	$a = \sqrt{bb + \sqrt{\frac{1}{2}b^4}} = 1,30656296, \&c. = CA$
Then		13	$aa - \frac{1}{4}bb = \square CP$, viz. $\square CH - \square HP = \square CP$
13	w^2	14	$\sqrt{aa - \frac{1}{4}bb} = 1,20710678, \&c. = CP$

From hence 'twill be easy to find the *Area* of any *Octagon*.

PROBLEM IX.

The Side of any regular Octagon being given, to find its Area:

Example. Suppose the Side given to be 12 Inches long; *First*, as $1 : 1,20710678 :: 12 : 14,48528136$ = the *Radius* of its inscrib'd Circle; then $12 \times 4 = 48$ is half the Sum of its Sides, and $48 \times 14,48528136 = 695,2935$ the *Area* required.

Fourthly, For a *Decagon*.

The Side of any regular Decagon (viz. a Polygon of ten equal Sides) is in Proportion to the Radius of

Its { *Circumscribing Circle*, as $1 : 1,61803398, \&c.$
Inscrib'd Circle as $1 : 1,53884176, \&c.$

Viz. { $BA : CA :: 1 : 1,61803398$
 $BA : CP :: 1 : 1,53884176$

Demonstration.

I. et { $b = BA = 1. a = CA$
 $c = DB$, and $y = Bx$

Then 1 $2a : b :: e : y$
 That is, $DA : BA :: DB : Bx$

1 $\therefore$ 2 $\begin{cases} 2ay = be \\ \text{and } 2y = \frac{be}{a} \end{cases}$

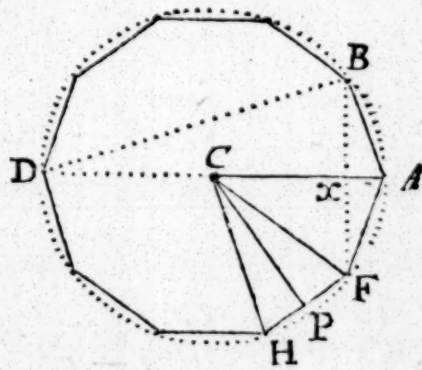
But 3 $2y : e :: 1 : 1,61803398$. See *Pentagon*.

3 $\therefore$ 4 $\frac{1e}{1,61803398} = 2y = \frac{be}{a} = \frac{1e}{a}$

4 $\div 1e$ 5 $1,61803398 = a = CA$

Again 6 $\begin{cases} aa - \frac{1}{4}bb = \square CP. \\ \text{viz. } \square CF = \square PF = \square CP. \text{ By Theorem II.} \end{cases}$

That is, 7 $\sqrt{2,61803398 - 0,25} = 1,53884176 = CP$



PROBLEM X.

The Side of any regular Decagon being given, to find its Area.

Example. Let the given Side be 14 Inches long; then, as
 $1 : 1,53884176 :: 14 : 21,543784$ = the Radius of the inscrib'd
 Circle; and $14 \times 5 = 70$ is half the Sum of its Sides. Lastly,
 $21,543784 \times 70 = 1508,06488$ the Area requir'd.

Fifthly, For a Dodecagon.

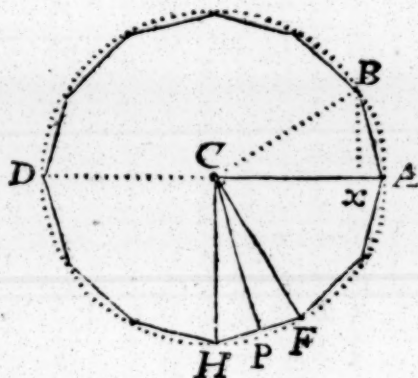
The Side of any regular Dodecagon (viz. a Polygon of twelve equal
 Sides) is in Proportion to the Radius of

Its { Circumscribing Circle, as 1 : to 1,93185165, &c.
 { Inscrib'd Circle as 1 : to 1,86632012, &c.
 Viz. { $BA : CA :: 1,03185165$
 { $BA : CP :: 1,86632012$

Demonstration.

Let $b = BA = 1$. $a = CA$ as before
 And $e = xA$; then $a - e = Cx$

First	1	{ $bb - \square Bx = ee$
		{ By Figure.
But	2	$Bx = \frac{1}{2} CA = \frac{1}{2} a$
2 $\odot^2$	3	$\square Bx = \frac{1}{4} aa$
1,	4	$bb - \frac{1}{4} aa = ee$
4 w^2	5	$\sqrt{bb - \frac{1}{4} aa} = ee$
Again	6	$aa - \frac{1}{4} aa = aa - 2ae + ee$
Viz.		$\square CB - \square Bx = \square Cx$
5 $\times$	7	$2a \sqrt{bb - \frac{1}{4} aa} = 2ae$
4 —	8	$bb - \frac{1}{4} aa - 2a \sqrt{bb - \frac{1}{4} aa} = ee - 2ae$
7,	9	$aa - \frac{1}{4} aa = aa + bb - \frac{1}{4} aa - 2a \sqrt{bb - \frac{1}{4} aa}$
9 $\div$	10	$2a \sqrt{bb - \frac{1}{4} aa} = bb$
10 $\odot^2$	11	$4bbaa - aaaa = b^4$
11 $\div$	12	$aaaa - 4bbaa = -b^4$
13, $C \square$	13	$aaaa - 4bbaa + 4b^4 = 3b^4 = 3$
1 w^2	14	$aa - 2bb = \sqrt{3} = 1,7320508075$
14 $\div$	15	$aa = 2bb + \sqrt{3} = 3,7320508075$
15 w^2	16	$a = \sqrt{3,7320508075} = 1,93185165 = CA$
Again	17	$aa - \frac{1}{4} bb = \square CP$. viz. $\square CF - \square PF = \square CP$
17, Hence	18	$CP = \sqrt{aa - \frac{1}{4} bb} = 1,86632012$. Q. E. D.



Conseſſary.

Hence if the Side of any *regular Dodecagon* be given, the *Radius* of its *inſcrib'd Circle* may be eaſily obtain'd, and thence the *Area* found; as in the laſt Problem.

The Work of the 'foregoing *Polygons*, being well conſider'd, will help the young *Geometer* to raiſe the like Proportions for others, if his Curioſity requires them: And not only ſo, but they will alſo help to form a true Idea of a Circle's *Periphery* and *Area*, according to the Method which I ſhall lay down in the next Chapter for finding them both.

C H A P. VI.

A new and eaſy Method of finding the Circle's Periphery and Area to any aſſign'd Exaſtneſs (or Number of Figures) by one Equation only. Alſo a new and facile Way of making Natural Sines and Tangents.

LET us ſuppoſe (*what is very eaſy to conceive*) the Circle's *Area* to be compos'd or made up of a vaſt Number of plain *Iſoſceles Triangles*, having their *acuteſt Angles* all meeting in the Circle's Center. And let us imagine the *Baſes* of thoſe *Triangles* ſo very ſmall, that their Sides and their Perpendicular Heights, *viz.* the *Radius's* of their circumscrib'd and inſcrib'd Circles (*vide* Problem 6.) may become ſo very near in *Length* to each other, as that they may be taken one for another without any ſenſible Error: Then will the *Peripheries* of their circumscribing and inſcrib'd Circles become (*altho' not co-incident, yet*) ſo very near to each other, as that either of them may be indifferently taken for one and the ſame Circle.

But how to find out the Sides of a *Polygon* (*viz.* the *Baſes* of thoſe *Iſoſceles Triangles*) to ſuch a convenient Smallneſs as may be neceſſary to determine and ſettle the Proportion betwixt a Circle's *Diameter* and its *Periphery* (*to any aſſign'd Exaſtneſs*) hath hitherto been a Work which requir'd great *Care* and much *Time* in its Performance; as may eaſily be conceived from the *Nature* of the *Method* uſ'd by all thoſe who have made any conſiderable Progreſs in it, *viz.* *Archimedes*, *Snellius*, *Hugenius*, *Mœtius*, *Van Culen*, &c. Theſe proceeded with the biſecting of an *Arch*, and found the *Value* of its *Chord* to a convenient Number of Figures

at every single Bisection, repeating their *Operations* until they had approach'd to the *Chord* design'd.

And this *Method* is made Choice of by the learned Dr. *Wallis* in his Treatise of *Algebra*; wherein, after he hath given us a large Account of the different Enquiries made by several (*very eminent in Mathematical Sciences*) in order to find out some easier and more expeditious Way of approaching to the Circle's *Periphery*, as in *Chap.* 82, 84, 85, 86, and several other Places, he comes to this Result, (*Page* 321.)

" 'Tis true, *saith he*, we might in like Manner proceed by continual Trisection, Quinisection, or other Section, if we had for these as convenient Methods of Operation as we have for Bisection: But because *Euclid* shews how to bisect an Arch Geometrically, but not to trisect, &c. and the one may be done (*Algebraically*) by resolving a Quadratick Equation, but not those other, without Equations of a higher Composition, I therefore make Choice of a continual Bisection, &c."

And then he lays down these following Canons:

The Subtense of $\frac{1}{6}$	1	into 6
of $\frac{1}{2}$	$\sqrt{2} - \sqrt{3}$	into 12
of $\frac{1}{4}$	$\sqrt{2} + \sqrt{2} + \sqrt{3}$	&c. 24
of $\frac{1}{8}$	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{3}$	48
of $\frac{1}{16}$	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3}$	96
&c.	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3}$	192
	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3}$	384
	$\sqrt{2} - \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{2} + \sqrt{3}$	768
	&c.	&c.

How tedious and troublesome the Work of these complicated Extractions is, I leave to the Consideration of those, who either have had Experience therein, or out of Curiosity will give themselves the Trouble of making Trial.

Again, in *Page* 347, the Doctor inserts a particular Method proposed by *Libnitius*, publish'd in the *Acta Eruditorum* at *Leipsick*, for the Month of *February* 1682, in order to find the Circle's *Area*, and consequently its *Periphery*, which is this:

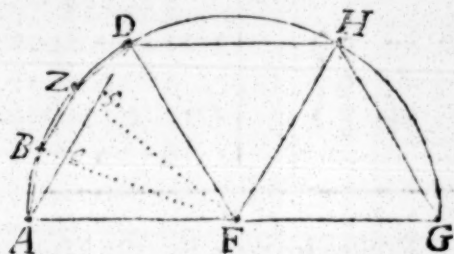
As 1 : to $\frac{1}{2} - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \frac{1}{13} - \frac{1}{15} + \frac{1}{17} - \frac{1}{19}$, &c. infinitely :: so is the Square of the *Diameter* to the Circle's *Area*. But this convergeth so very slowly, that it is not worth the Time to pursue it.

I shall here propose a new Method of my own, whereby the Circle's *Periphery*, and consequently its *Area*, may be obtain'd infinite-

infinitely near the Truth, with much greater Ease and Expedition than either that of *Bisection*, or that of *Libnitius*, as above, or any other Method that I have yet seen ; it being perform'd by *resolving* only one *Æquation*, deduced by an easy Process from the Property of a Circle, (*known to every Cooper*) which is this :

The Radius of every Circle is equal to the Chord of one sixth Part of its Periphery. That is, $AD = DH = HG$, the Chords of one third Part of the Semicircle, are each equal to AF its Radius. Then if the Arch AD be trisected, it will be $AB = BZ = ZD$.

Let $\begin{cases} R = AF = 1 \\ c = AD = 1 \\ a = AB. \end{cases}$ Quære a .



Then	1	$R : a :: a : \frac{aa}{R} = Be$
And	2	$R : a :: R - \frac{aa}{R} : c - 2a$
That is,	3	$FB : BZ :: Fe : ex = AD - 2a$
For		$\triangle AFB$, and $\triangle BAe$, are alike.
		And $AB = Ae = Dx$, &c.
2	4	$Rc - 2Ra = Ra - \frac{aaa}{R}$
4 × &c.	5	$3R^2a - aaa = RRc$. That is, $3a - aaa = 1$
		Here a = the Chord of $\frac{1}{18}$ Part of the Circle.
		For $\frac{1}{3}$ of $\frac{1}{6} = \frac{1}{18}$

Next, To trisect the Arch AB .

Let	1	$3y - y^3 = a$ the last Chord.
1 $\odot^5$	2	$27y^3 - 27y^5 + 9y^7 - y^9 = a^3$
1 $\times$ 3	3	$9y - 3y^3 = 3a$
3 $-$ 2	4	$9y - 30y^3 + 27y^5 - 9y^7 + y^9 = 3a - a^3 = 1$
		Here y = the Chord of $\frac{1}{54}$ Part of the Circle.

Again, To trisect the Arch whereof y is the Chord.

Let	1	$3a - a^3 = y$
1 $\odot^3$	2	$27a^3 - 27a^5 + 9a^7 - a^9 = y^3$
1 $\odot^5$	3	$243a^5 - 405a^7 + 270a^9 - 90a^{11} + 15a^{13} - a^{15} = y^5$
		1 $\odot^7$

I	⊙ ⁷	4	{	2187a ⁷ — 5103a ⁹ + 5103a ¹¹ — 2835a ¹³ +	
			}	945a ¹⁵ = y ⁷	
I	⊙ ⁹	5	{	19683a ⁹ — 59049a ¹¹ + 78732a ¹³ —	
			}	61236a ¹⁵ = y ⁹	
I	×	9	6	27a — 9a ³ = 9y	
2	×	30	7	810a ³ — 810a ⁵ + 270a ⁷ — 30a ⁹ = 30y ³	
3	×	27	8	{	6561a ⁵ — 10935a ⁷ + 7290a ⁹ — 2430a ¹¹ +
			}	405a ¹³ + 27a ¹⁵ = 27y ⁵	
4	×	9	9	{	19683a ⁷ — 45927a ⁹ + 45927a ¹¹ —
			}	25515a ¹³ + 8505a ¹⁵ = 9y ⁷	
6	—	7	10	{	27a — 819a ³ + 7371a ⁵ — 30888a ⁷ +
+ 8	—	9		}	72930a ⁹ — 107406a ¹¹ +
+ 5				}	104652a ¹³ — 69768a ¹⁵
					Here a = the Chord of $\frac{1}{108}$ Part of the Circle.

Proceeding on in this Method of continually trisecting the Arch of every new Chord, and still connecting the produced *Æquations* into one, as in the *two last Trisections*, 'twill not be difficult to obtain the *Chord* of any assign'd Arch, how small soever it be.

Now, in order to facilitate the Work of raising these *Æquations* to any considerable Height, 'twill be convenient to add a few useful Observations concerning their Nature, and of such Contractions as may be safely made in them; which, being well understood, will render the Work very *easy*.

1. I have observ'd, that every Trisection will gain or advance one Figure in the Circle's Periphery, but no more. Therefore so many Places of Figures as are at first design'd to be perfect in the Periphery, so many Trisections must be repeated to raise an *Æquation* that will produce a Chord answerable to that Design.

2. I have also found, that all the superior Powers (of a) whose Indices are greater than the Number of Trisections, (viz. whose Indices are greater than the Number of design'd Figures) may be wholly rejected as insignificant.

3. When once the Number of Trisections, and thence the highest Power (of a) is determined, the third Process (viz. the third Trisection) may be made a fix'd or constant Canon; for by it, and Multiplication only, all the succeeding Trisections (how many soever they are) may be completed without repeating the several Involutions.

4. In raising and collecting the Co-efficients of the several Powers (of a) 'twill be sufficient to retain only so many significant Figures (at a^3) as there is designed to be Places of Figures in the Periphery (or at most but two more) and every succeeding superior Power may be allow'd to decrease two Places of significant Figures: But herein great Care must be taken to supply the Places, of those Figures that are omitted, with Cyphers, that so the whole and exact Number of Places may be truly adjusted; otherwise all the Work will be erroneous.

Now the Number of those supplying Cyphers may be very conveniently denoted by Figures placed within a Parenthesis, thus: $576(8)a^3$, may signify $57600000000a^3$, as in the following *Æquations*. The like may be done with Decimal Parts, thus: $(.7)658$ may signify $.0000000658$, &c. which will be found very useful in the Solution of these and the like *Æquations*.

The aforesaid Contractions may be safely made, because both the superior Powers of a , which are rejected; as also those Numbers that are omitted in the Co-efficients (and supply'd with Cyphers) would produce Figures so very remote from Unity, as that they would not affect the *Chord* design'd; that is, they would not affect the *Chord* in that Place wherein the design'd *Periphery* is concerned; as will in Part appear in the following Example.

If these Directions be carefully minded, 'twill be easy to raise an *Æquation* that will produce the Side of a regular Polygon, whose Number of Sides shall be vastly numerous, consequently infinitely small: But, I presume, 'twill be sufficient for an Example to find the Side of a Polygon consisting of 258280326 equal Sides; that is, if I find the *Chord* of $\frac{1}{258280326}$ Part of the Circle's Periphery, and that requires but sixteen *Trisections*, which being order'd, as before directed, will produce this *Æquation*.

$$\left\{ \begin{array}{l} 43046721a - 332360179486968612(4)a^3 \\ + 769837653199714(20)a^5 - 8491218532841(35)a^7 \\ + 54633331143(50)a^9 - 230083348(66)a^{11} \\ + 6830988(79)a^{13} - 15072(94)a^{15} \end{array} \right\} = 1$$

Here the Value of a will have 23 Places of Figures true; that is, the Sides of the inscrib'd and circumscrib'd Polygons will be exactly the same to 23 Places of Decimal Parts, but no farther; all which may be easily obtain'd at two Operations. And for the first, 'twill be sufficient to take only three Terms of the *Æquation*, which will admit of being yet farther contracted, thus:

Let

$$\text{Let } \left\{ \begin{array}{l} 43046721a - 3323601794(12)a^3 \\ + 76983765(27)a^5 \end{array} \right\} = 1$$

And let $r + e = a$; then rejecting all the Powers of e , that arise by Involution above eee ,

$$\text{It will be } r^3 + 3rre + 3ree + eee = aaa$$

$$\text{And } r^5 + 5r^4e + 10r^3ee + 10rree + eee = a^5$$

Then the first single Value of r may be thus found:

$$43046721) 1,00000000 \text{ (},00000002 = r$$

This ,00000002 = r being duly involv'd, and its Powers multiply'd into their respective Co-efficients, will produce

$$\left. \begin{array}{l} +,86093441 + 43046721e \\ -,02658881 - 3988322e - 199416(9)ee - 3324(18)eee \\ +,00024635 + 61587e + 6159(9)ee + 308(18)eee \end{array} \right\} = 1$$

$$\text{viz. } ,83459196 + 39119986e - 193257(9)ee - 3016(18)eee = 1$$

$$\text{Hence } 39119986e - 193257(9)ee - 3016(18)eee = 0,16540804$$

All the Terms of this last Equation being divided by 193257(9) the Co-efficient of ee , it will then become

$$,0000002024e - ee - ,156(5)eee = ,00000000000000008558968 = D$$

$$\text{Consequently, } \left\{ \frac{D + 156(5)eee}{,0000002024 - e} = e \right.$$

Operation.

$$\begin{array}{r} ,0000002024) \quad ,00000000000000008558968 \quad (,000000004 = e \\ - e \quad ,0000000043 : + ,000000000000000009984 \quad = 156(5)eee \end{array}$$

$$1 \text{ Di. } ,000000198) \quad ,00000000000000008568952 \quad (,000000004327$$

$$2 \text{ Di. } ,0000001981$$

792

6489

5943

$$\text{First } r = ,00000002$$

5465

3962

$$+ e = ,000000004327$$

&c.

$$r + e = ,000000024327 = a.$$

Or rather new r for a second Operation.

Now, if this first Value of $a = ,000000024327$ were not continued to more Places of Figures by a second Operation, but only multiply'd into the Number of Chords, viz. ,000000024327 $\times$ 258280326 = 6,28318539, &c. the Periphery of that Circle whose Diameter is 2, nearer than either Archimedes, or Mæcius's Proportion: For

Archimedes makes it 6,285714, &c. viz. As 7 to 22. And Mœtius makes it 6,28318584, &c. viz. As 113 to 355.

But if the whole *Æquation* before propos'd be now taken, and we proceed to a second *Operation*, the Value of *a* may be increas'd with twelve Places of *Figures* more, and those may be obtain'd by plain *Division* only.

Thus, let $r + e = a$, as before, and let all the Powers of *e* be now rejected as insignificant;

Then will $\left\{ \begin{array}{l} r + e = a \\ r^3 + 3r^2e = a^3 \\ r^5 + 5r^4e = a^5 \\ r^7 + 7r^6e = a^7 \end{array} \right\}$ and $\left\{ \begin{array}{l} r^9 + 9r^8e = a^9 \\ r^{11} + 11r^{10}e = a^{11} \\ r^{13} + 13r^{12}e = a^{13} \\ r^{15} + 15r^{14}e = a^{15} \end{array} \right\}$

The several Powers of $r = ,000000024327$ being rais'd, and multiply'd into their respective *Co-efficients*, will produce these following Numbers:

+ 1,047197581767	+ 43046721e	}	= 1
— ,047849196598394865	— 5900751e		
+ ,000655906484595355	+ 134810e		
— ,000004281440413375	— 1232e		
+ ,000000016302517803	+ 6e		
— ,000000000040631167	— 0e		
+ ,00000000000071388	+ 0e		
— ,00000000000000093	— 0e		

Viz. $1,000000026474745106 + 37279554e = 1$

Hence $37279554e = - ,000000026474745106 = D$: Or rather $- 37279554e = ,000000026474745106 = D$

Consequently, $\left\{ \frac{D}{37279554} = -e \right.$

Operation.

$37279554) ,000000026474745106 ((,15)710167967 = -e$
260956878

37905730

37279554

62617660

37279554

&c.

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(viz. $\sqrt{2 + \sqrt{3}}$) must have 99 Places of *Figures* in it. The third Root (viz. $\sqrt{2 + \sqrt{2 + \sqrt{3}}}$) must have 96 Places in it, &c. every *Extraction* being allow'd to decrease three Places, that so the last Root (viz. the Chord sought) may consist of 24 Places of *Figures*, as above.

I say, whoever duly considers the Trouble of these so often repeated *Extractions* will, I presume, be pleas'd with what I have done. For truly, when I consider the great Time and Care required in them, I cannot but admire at the Patience of the laborious *Van Culen*, who proceeded that Way until he had found the Circle's Periphery to Thirty-six Places of *Figures*, to wit, 6,28318530717958647692528676655900576. These Numbers are said to be engraven upon his Tomb-Stone in St. Peter's Church in Leyden, for a Memorial of so great a Work.

Having thus obtain'd the Circle's Periphery, its Arch may easily be found (to the same Number of *Figures*) by Problem 6. That is, if Half the Periphery of any Circle be multiply'd into Half its Diameter, the Product will be that Circle's Area, as will appear farther on. Therefore 3,141592653589793 will be the Area of the Circle whose Diameter is 2.

Thus I have shew'd the young Geometer how to find the Circle's Periphery and Area to what Exactness he pleases to approach; for precisely true it cannot be found, notwithstanding the late Pretensions of a certain Frenchman who hath published to the World (in the Works of the Learned) that after twenty-five Years Study he had found the Quadrature of the Circle: But if he had perus'd the 83d Chapter of Dr. Wallis's Algebra, he might there have seen his Error, viz. the Impossibility of what he pretended to; for it is as impossible to square the Circle (that is, to find its true Area) as it is to find the Root of a Surd Number.

Note, What I have here propos'd and done by the Trisection of an Arch, may as easily and much more speedily be perform'd by Quinquesecion or Septisection, &c. But because the Scheme for Trisection is more simple, and may be easier understood by a Learner than those of the other Sections (of which see my Compendium of Algebra, Pages 76 and 79) I have for that Reason made Choice of Trisection.

As to the Proportion of one Circle to another, and of the Circle to the Ellipsis, &c. those shall be fully shew'd when we come to the Fifth Part.

Before I conclude this Part, I shall make some *Use* or *Application* of the above-found *Periphery*, in finding the *Quantity* of *Angles*, which is done by the Help of *Right-lines*, call'd *Sines* and *Tangents*, the Length whereof are calculated to every Degree and Minute of a *Quadrant*, by much Labour. But I shall here shew how to find the natural *Sine* (and consequently the natural *Tangent*) of any propos'd *Arch* or *Angle*, by two *Equations*, without the Help of any *precedent Sine*, as usual; which I did some Years ago communicate to the ingenious Mr. *Joseph Ralphson*, and he so well approv'd of them as to make them the 20th and 21st *Problems* in the *second Edition* of his *Analysis Aequationum Universalis*.

And because, in finding the *Quantity* of *Angles*, every *Circle* is suppos'd to be divided into 360 equal *Parts*, call'd *Degrees*; every *Degree* is subdivided into 60 *Parts*, call'd *Minutes*; and every *Minute* into 60 *Seconds*, &c. (See Page 294.)

Therefore 360) 6,2831853, &c. (0,0174532925, &c. is an *Arch* of the above-found *Periphery*, equal to the *Arch* of one *Degree*.

And 60) 0,0174532925, &c. (0,0002908882, &c. = the *Arch* of one *Minute*.

Then if the given *Arch* (or *Angle*) be less than 45 *Degrees*, reduce it into *Minutes*, and multiply those *Minutes* into this constant *Multiplicator*, viz. 0,0002908882 calling the *Product* *p*. And for the *Sine* sought put *a*. Then it will be — *aaaa* + 12*p**aaa* — 195*aa* — 36*pp**aa* + 240*p**a* = 45*pp*.

Example.

Let it be required to find the *Sine* of $19^{\circ}. 13' = 1153'$. Here $0,0002908882 \times 1153 = 0,3353940946 = p$. And — $a^4 + 4,024729a^3 - 199,049611aa + 80,494583a = 5,06201394$.

Let $r + e = a$

$$\text{Then } \begin{cases} rr + 2re + ee = aa \\ rrr + 3rre + ee = aaa \\ rrrr + 4rrre + 6rree = aaaa \end{cases}$$

Note, In this Case the first *r* may always be taken equal to the first *Figure* in the *Product* = *p*. Viz. here $r = 0,3$ which being involved as its *Powers* direct, and those *Powers* multiply'd into the respective *Co-efficients* of the *Equation*; it will be

$$\left\{ \begin{array}{l} + 24,1483 + 80,49e \\ - 17,9144 - 119,43e - 199,05ee \\ + 0,1086 + 1,08e + 3,62ee \\ - 0,0081 - 0,11e - 0,54ee \end{array} \right\} = 5,06201394$$

$$\text{Viz. } 6,3344 - 37,97e - 135,97ee = 5,06201$$

Hence

Hence $37,97ee + 195,97ee = 1,27239$

And $0,123e + ee = 0,006492 = D$

THEOREM $\left\{ \frac{D}{,193 + e} = e \right.$

Operation. $0,193 \overline{) 0,006492} \quad (0,029 = e$

$+ e = ,029 \quad 42$

1. Divisor $,21 \quad 2292$

1998

2. Divisor $,222$

First $r = 0,3$

$+ e = 0,029$

$r + e = 0,329 = r$ for a second Operation.

Which being involv'd and multiply'd, &c. as before, will produce these Numbers:

$+ 26,48271781 + 80,49458e$

$- 21,54532894 - 130,97464e - 199,0496ee$

$+ 0,14332578 + 1,30692e + 3,9724ee$

$- 0,01171611 - 0,14244e - 0,6494ee$

Viz. $5,06899854 - 49,31558e - 195,7266ee = 5,06201394$

Hence $49,31558e + 195,7266ee = ,0069846$; which being divided by $195,7266$ the Co-efficient of ee , will become $,25196e + ee = ,0000356854 = D$

Then $\left\{ \frac{D}{,25196 + e} = e \right.$

Operation. $0,25196 \overline{) ,0000356854} \quad (0,0001415 = e$

$+ e = 0,00014 \quad 2520$

1. Divisor $0,2520 \quad 104854$

2. Divisor $0,25210 \quad 100840$

40140

25210

Last $r = 0,329$

&c.

$+ e = 0,0001415$

$r + e = a = 0,3291415$ being the natural Sine of $90^\circ. 13'$. As was required.

Thus you may find the Right Sine of any Arch or Angle less than 45 Degrees. But,

But, if the given *Arch* be greater than 45 *Degrees*, you must take its Complement to 90°. viz. subtract it from 90 *Degrees*, and reduce the Remainder into *Minutes*, as before. Then multiply the *Square* of these *Minutes* into this constant *Multiplicator*, 0,000000084616 calling their *Product* p , and putting a = the *Sine* sought, as before. Then will $a^4 + 28a^3 + 195aa + 36paa + 108pa - 28a = 196 - 81p$.

Example.

Suppose it were required to find the *Sine* of 75°. 32'. or (which is the same Thing) to find the *Co-sine* of 14°. 28'. = 868', whose *Square* 753424 $\times$ 0,000000084616 = 0,06375172518 = p . Hence the *Æquation* in *Numbers* will be $aaaa + 28aaa + 197,295062aa - 21,114814a = 190,8361102588$.

$$\text{Let } r - e = a \quad \text{And } r = 1$$

$$\text{Then } \begin{cases} rr - 2re + ee = aaa \\ rrr - 3rre + 3ree = aaa \\ rrrr - 4rrre + 6rree = aaaa \end{cases}$$

Note, I here take $r = 1$ because the *Arch* is so near to 90°. and therefore I make it $r - e = a$.

$$\text{Then } \left\{ \begin{array}{l} - 21,1148 + 21,11e + \\ + 197,2956 - 394,59e + 197,29ee \\ + 28,0000 - 84,00e + 84,00ee \\ + 1,0000 - 4,00e + 6,00ee \end{array} \right\} = 190,8361$$

$$\text{Viz. } 205,1808 - 461,48e + 287,29ee = 190,8361$$

$$\text{Hence } 461,48e - 287,29ee = 14,3447$$

$$\text{And } 1,606e - ee = ,049930 = D$$

$$\text{THEOREM } \left\{ \frac{D}{1,606 - e} = e \right.$$

$$\text{Operation. } \begin{array}{l} 1,606 \quad ,049930 \quad (0,031 = e \\ - e = 0,031 \quad 471 \end{array}$$

$$1. \text{ Divisor } \begin{array}{r} 1,57 \quad 2830 \\ \hline \end{array}$$

$$2. \text{ Divisor } \begin{array}{r} 1,575 \quad 1575 \\ \hline \end{array}$$

&c.

$$\text{First } r = 1,000$$

$$- e = 0,031$$

$$r - e = 0,969 = r$$

for a second *Operation*; which, being involv'd as before, will produce these following *Numbers*;

$$\begin{array}{r}
 - 20,460254766 + 21,11481e \\
 + 185,252368710 - 382,35783e + 197,2951ee \\
 + 25,475889852 - 78,87272e + 81,5960ee \\
 + 0,881647759 - 2,63941e + 5,6337ee
 \end{array}$$

$$\begin{array}{r}
 \text{Viz. } 191,149651515 - 443,75515e + 284,5248ee \\
 = 190,836110259
 \end{array}$$

$$\begin{array}{r}
 \text{Hence it will be } 443,75515e - 284,5248ee = 0,313541256 \\
 \text{And } 1,55963e - ee = ,0011019821 = D
 \end{array}$$

$$\text{Then } \left\{ \frac{D}{1,55963 - e} = e \right.$$

$$\begin{array}{r}
 \text{Operation. } 1,55963) 0,0011019821 \text{ (} 0,0007068 = e \\
 - e = 0,00070 \quad 109123
 \end{array}$$

$$\begin{array}{r}
 1. \text{ Divisor } 1,5589 \quad 1075210 \\
 \quad \quad \quad 935358 \\
 2. \text{ Divisor } 1,55893 \quad 1398520 \\
 \quad \quad \quad 1247144, \text{ \&c.}
 \end{array}$$

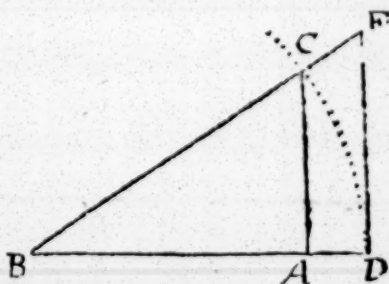
$$\begin{array}{r}
 \text{Last } r = 0,969 \\
 - e = 0,0007068
 \end{array}$$

$$r - e = a = 0,9682932 \text{ the Sine of } 75^\circ. 32'. \text{ as was required.}$$

Having found the *Sine* and *Co-sine* of any *Arch*, the *Tangent* is usually found by this *Proportion* :

$$\text{Viz. } \left\{ \begin{array}{l} \text{As the Co-sine of any Arch : is to the Sine of that Arch : : so is} \\ \text{the Radius : to the Tangent of the same Arch.} \end{array} \right.$$

For supposing $BC = BD$ Radius, AC the *Sine* of the *Arch* CD . Then BA is the *Co-sine*, and FD the *Tangent* of the same *Arch*. But $BA : CA : : BD : FD$, &c. Now by this *Proportion* there is required to be given both the *Sine* and *Co-sine* of the *Arch*, to find the *Tangent*. 'Tis true, if the *Radius*, and either the *Sine* or the *Co-sine* be given, the other may be found, thus, $\sqrt{BC^2 - CA^2} = BA$. Or $\sqrt{BC^2 - BA^2} = CA$. But, if either the *Sine* or *Co-sine* be given, the *Tangent* may (I presume) be more easily found by the following *Theorems* :



Let $BC = 1$. $CA = S$. $BA = x$ and $FD = T$. Then, if S be given, T may be found by this

$$\text{THEOREM} \left\{ \sqrt{\frac{SS}{1-SS}} = T \right.$$

Or if x be given, T may be found by this

$$\text{THEOREM} \left\{ \sqrt{\frac{xx}{1-xx}} = T \right.$$

Let the *Sine* of $90^\circ. 13'$. (*before found*) be given, *viz.* $0,3291415 = S$, to find T the *Tangent* of the same *Arch*. First $0,3291415 \times 0,3291415 = 0,108334127 = SS$. Again $1 - 0,108334127 = 0,891665873 = 1 - SS$. Then $0,891665873 \div 0,108334127 = 0,1214963253$ and $\sqrt{0,1214963253} = 0,3485632 = T$, the *Tangent* of $19^\circ. 13'$. As was required. And so you may proceed to find $T =$ the *Tangent*, when $x =$ the *Co-sine* is given.

Perhaps it may here be *expected*, that I should have shew'd and *demonstrated* (or at least have inserted) the *Proportions* from whence the foregoing *Equations* for making *Sines* were produced; but I have omitted that, as also their *Use* in computing the *Sides* and *Angles* of plain *Triangles* by the *Pen* only (*viz. without the Help of Tables*) for the *Subject* of my *Discourse* hereafter, if *Health* and *Time* permit.

In the mean *Time*, what is here done may suffice to shew, that the making of *Sines* by such a *laborious* and *operose* Way, as was formerly used, is in a great *Measure* overcome; which, I think, I may justly claim as my own.

A N

INTRODUCTION

TO THE

Mathematicks.

PART IV.

CHAP. I.

Definitions of a Cone, and its Sections.

THERE are several Definitions given of a *Cone*: The Learned Dr. *Barrow*, upon *Euclid*, hath it thus:

“ A *Cone* (*saieth he*) is a Figure made when one Side of
 “ a Rectangle Triangle, (*viz.* one of those Sides that contain the
 “ Right Angle) remaining fix'd, the Triangle is turn'd round
 “ about, 'till it return to the Place from whence it first moved:
 “ And if the fix'd Right Line be equal to the other which con-
 “ taineth the Right Angle, then the Cone is a Rectangled Cone:
 “ but if it be less, 'tis an Obtuse-angled Cone; if greater, an Acute-
 “ angled Cone. The *Axis* of a Cone is that fix'd Line about which
 “ the Triangle is mov'd: The *Base* of a Cone is the Circle, which
 “ is describ'd by the Right Line mov'd about.”

“ (*Defin.* 18, 19, 20. *Euclid.* 11.)

Sir *Jonas Moor*, in his Treatise of *Conical Sections* (taken out of the Works of *Mydorgius*) defines it thus:

“ If a Line of such a Length as shall be needful shall, upon a
 “ Point fix'd above the Plain of a Circle, so move about the Cir-
 “ cle, until it return to the Point from whence the Motion began,
 “ the Superficies that is made by such a Line is call'd a *Conical*
 “ *Superficies*; and the solid Figure contain'd within that Superficies
 “ and the Circle is call'd a *Cone*. The Point remaining still is the
 “ Vertex of the Cone, &c.”

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Altho' both these *Definitions* are equally true, and, with a little Consideration, may be pretty easily understood; yet I shall here propose one very different from either of them; and, as I presume, more plain and intelligible, especially to a *Learner*.

If a Circle describ'd upon stiff Paper (or any other pliable Matter) of what Bigness you please, be cut into two, three, or more *Sectors*, either equal or unequal, and one of those *Sectors* be so roll'd up, as that the *Radii* may exactly meet each other, it will form a *Conical Superficies*.

That is, if the *Sector HVG* be cut out of the Circle, and so roll'd up as that the *Radii VH* and *VG* may just meet each other in all their Parts, it will form a *Cone*, and the Center *V* will become a *Solid Point*, call'd the *VERTEX* of the *Cone*; the *Radius VH*, being every-where equal, will be the Side of the *Cone*, and the Arch *HG* will become a Circle, whose *Area* is call'd the *Cone's Base*.

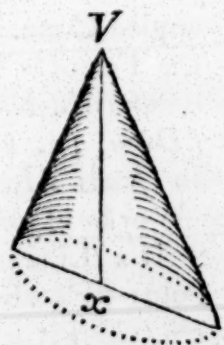
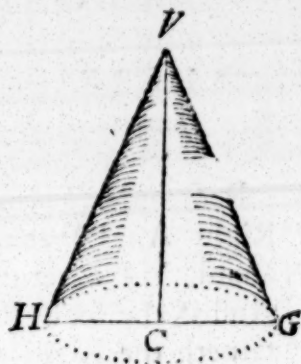
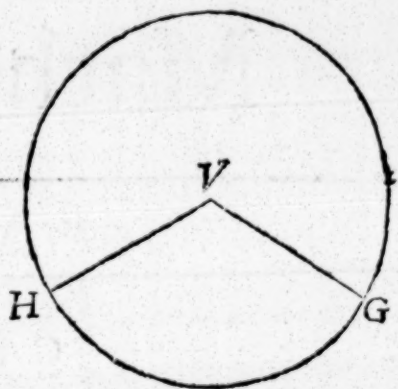
A *Right Line* being suppos'd to pass from the *Vertex*, or Point *V*, to the Center of the *Cone's Base*, as at *C*, that Line (viz. *VC*) will be the *AXIS*, or *perpendicular Height* of the *Cone*.

If a *Solid* be actually made in such a Form, it will be a compleat or perfect *Cone*; which I shall all along call a *Right Cone*, because its *Axis VC* stands at *Right Angles* with the Plain of its *Base HG*, and its *Sides* are every-where equal.

Any *Cone*, whose *Axis* is not at *Right Angles* with the Plain of its *Base*, may be properly call'd an *imperfect Cone*, because its *Sides* are not every-where equal (as in the annexed Figure.) Now, such an imperfect *Cone* is usually call'd a *Scalene*, or *Oblique Cone*.

Any solid *Cone* may be cut by Plains (which I shall all along hereafter call *Right Lines*) into five *Sections*.

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SECT. I.

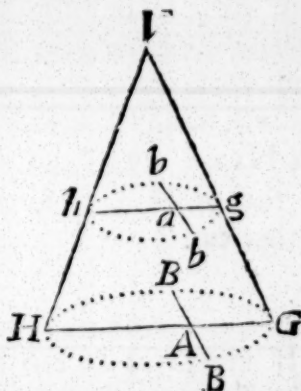
If a Right Cone be cut directly thro' its *Axis*, the *Plain* or *Superficies* of that *Section* will be a plain *Isoceles Triangle*, as *HVG* Fig. 2, viz. the Sides (*HV* and *VG*) of the Cone will be the Sides of the Triangle. the Diameter (*HG*) of the Cone's *Base* will be the *Base* of the Triangle, and (*VC*) its *Axis* will be the perpendicular *Height* of the Triangle.

SECT. 2.

If a Right Cone be cut (*any where*) off by a Right-line parallel to its *Base*, as *h g* (it will be easy to conceive, that) the *Plain* of that *Section* will be a Circle, because the Cone's *Base* is such : wherein one Thing ought to be clearly understood, which may be laid down as a *Lemma*, to demonstrate the Properties of the following *Sections*.

LEMMA. { If any two Right Lines, inscrib'd within a Circle, do cut or cross each other (as *h g* doth *b b* in the annexed Figure) the Rectangle made of the Segments of one of the Lines will be equal to the Rectangle made of the Segments of the other Line. (See Theorem 15. Page 315.)

That is, $h a \times g a = b a \times a b$ } &c.
And $H A \times G A = B A \times A B$ } &c.
consequently if $h a a = a b$, and if $B A = A B$, then it will be $h a \times g a = \square b a$, and in the Cone's *Base* $H A \times G A = \square B A$.



SECT. 3.

If a Right Cone be (*any where*) cut off by a Right Line that cuts both its *Sides*, but not parallel to its *Base* (as *TS* in the following Figure) the *Plain* of that *Section* will be an *Ellipsis* (*vulgarly called an Oval*) viz. an oblong or imperfect Circle, which hath several *Diameters*, and two particular *Centers*. That is,

1. Any Right Line that divides an *Ellipsis* into two equal Parts is call'd a *Diameter*; amongst which the *longest* and the *shortest* are particularly distinguish'd from the rest, as being of most general Use; the other are only applicable to particular Cases.

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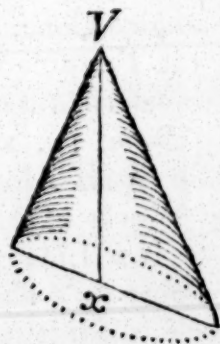
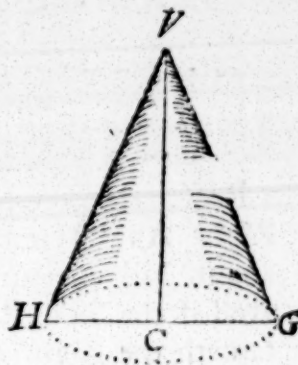
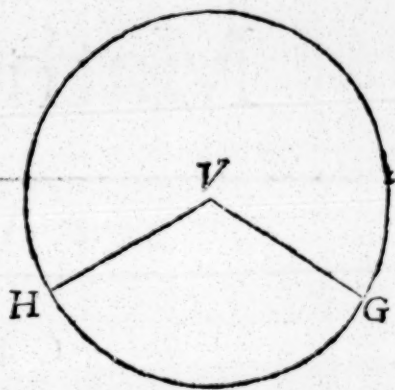
A *Right Line* being suppos'd to pass from the *Vertex*, or Point *V*, to the *Center* of the *Cone's Base*, as at *C*, that Line (viz. *VC*) will be the *AXIS*, or *perpendicular Height* of the *Cone*.

If a *Solid* be actually made in such a Form, it will be a compleat or perfect *Cone*; which I shall all along call a *Right Cone*, because its *Axis VC* stands at *Right Angles* with the Plain of its *Base HG*, and its *Sides* are every-where equal.

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SECT. 1.

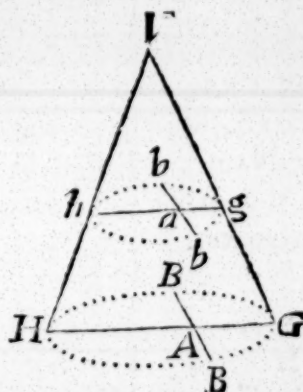
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SECT. 2.

If a Right Cone be cut (*any where*) off by a Right-line parallel to its *Base*, as *h g* (it will be easy to conceive, that) the *Plain* of that *Section* will be a *Circle*, because the Cone's *Base* is such: wherein one Thing ought to be clearly understood, which may be laid down as a *Lemma*, to demonstrate the Properties of the following *Sections*.

LEMMA. { If any two Right Lines, inscrib'd within a Circle, do cut or cross each other (as *h g* doth *b b* in the annexed Figure) the Rectangle made of the Segments of one of the Lines will be equal to the Rectangle made of the Segments of the other Line. (See Theorem 15. Page 315.)

That is, $h a \times g a = b a \times a b$ } &c.
And $H A \times G A = B A \times A B$ }
consequently if $h a = a b$, and if $B A = A B$,
then it will be $h a \times g a = \square b a$, and
in the Cone's *Base* $H A \times G A = \square B A$.



SECT. 3.

If a Right Cone be (*any where*) cut off by a Right Line that cuts both its *Sides*, but not parallel to its *Base* (as *TS* in the following Figure) the *Plain* of that *Section* will be an *Ellipsis* (vulgarly called an *Oval*) viz. an oblong or imperfect Circle, which hath several *Diameters*, and two particular *Centers*. That is,

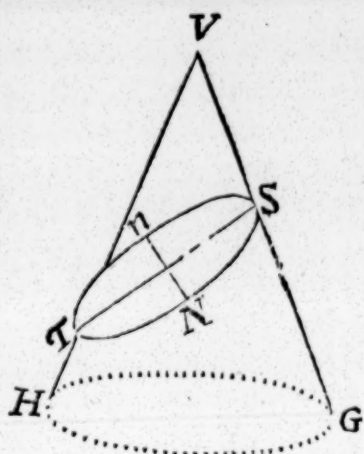
1. Any Right Line that divides an *Ellipsis* into two equal Parts is call'd a *Diameter*; amongst which the *longest* and the *shortest* are particularly distinguish'd from the rest, as being of most general Use; the other are only applicable to particular Cases.

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2. The longest *Diameter* (as TS) is called the *Transverse Diameter*, or *Transverse Axis*, being that *Right Line* which is drawn thro' the Middle of the *Ellipsis*, and doth shew or limit its Length.

3. The shortest *Diameter*, call'd the *Conjugate Diameter*, is a *Right Line* that doth intersect or cross the *Transverse Diameter* at *Right Angles*, in the Middle or common Center of the *Ellipsis* (as Nn) and doth limit the *Ellipsis's Breadth*.



4. The two *Points*, which I call particular *Centers* of an *Ellipsis* (for a Reason which shall be shew'd farther on) are two *Points* in the *Transverse Diameter*, at an equal Distance each Way from the *Conjugate Diameter*, and are usually call'd *NODES*, *FOCI*, or *burning Points*.

5. All *Right Lines* within the *Ellipsis* that are parallel to one another, and can be divided into two equal Parts, are called *ORDINATES* with Respect to that *Diameter* which divides them: And if they are parallel to the *Conjugate*, viz. at *Right Angles* with the *Transverse Diameter*, then they are call'd *Ordinates rightly apply'd*. And those two that pass through the *Foci* are remarkable above the rest, which, being equal and situated alike, are call'd both by one Name, viz. *LATUS RECTUM*, or *Right Parameter*, by which all the other *Ordinates* are regulated and valued; as will appear farther on.

Sect. 4.

If any Cone be cut into two Parts by a *Right-line* parallel to one of its Sides (as SA in the following Scheme) the Plain of that Section (viz. $SbBABbS$) is call'd a *PARABOLA*.

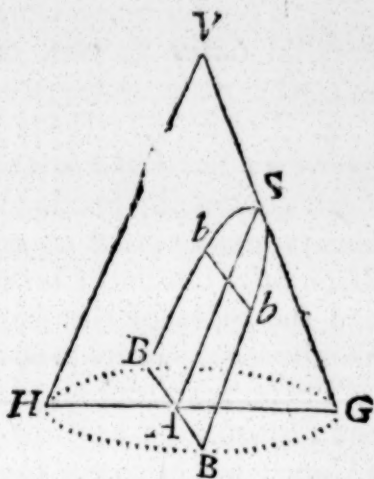
1. A *Right Line* being drawn thro' the Middle of any *Parabola* (as SA) is call'd its *Axis*, or intercepted *Diameter*.

2. All *Right Lines* that intersect or cut the *Axis* at *Right Angles* (as BB and bb are suppos'd to cut or cross SA) are call'd *Ordinates rightly apply'd* (as in the *Ellipsis*) and the greatest *Ordinate*, as BB , which limits the Length of the *Parabola's Axis* (SA) is usually call'd the *Base* of the *Parabola*.

3. That

3. That *Ordinate* which passes thro' the *Focus*, or burning Point of the *Parabola*, is called the *Latus Rectum*, or *Right Parameter* (as in the *Ellipsis*) because by it all the other *Ordinates* are proportion'd, and may be found.

4. The *Node*, *Focus*, or burning Point of the *Parabola*, is a Point in its *Axis* (but not a *Center*, as in the *Ellipsis*) distant from the *Vertex*, or Top of the Section, (viz, from *S*) just $\frac{1}{4}$ Part of the *Latus Rectum*; as shall be shewn farther on.

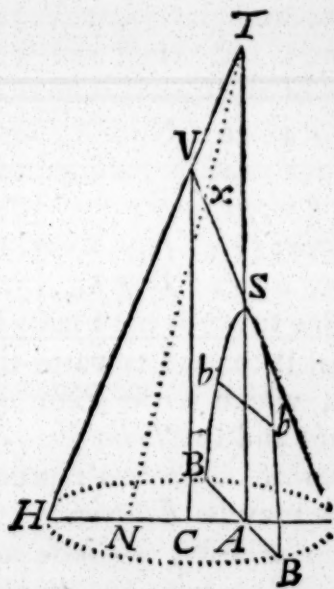


5. All *Right Lines* drawn within a *Parabola* parallel to its *Axis* are call'd *Diameters*; and every *Right Line*, that any of those *Diameters* doth bisect or cut into two equal Parts, is said to be an *Ordinate* to the *Diameter* which bisects it.

SECT. 5.

If a Cone be any where cut by a *Right Line*, either parallel to its *Axis* (as *SA*, or otherwise as *xN*) so as the cutting Line being continued thro' one Side of the Cone (as at *S* or *x*) will meet with the other Side of the Cone if it be continued or produced beyond the *Vertex V*, as at *T*; then the Plain of that Section (viz. the Figure *S b B B b S*) is call'd an *HYPERBOLA*.

1. A *Right Line* being drawn thro' the Middle of any *Hyperbola*, viz. within the Section (as *SA*, or *xN*) is call'd the *Axis* or intercepted *Diameter* (as in the *Parabola*) and that Part of it which is continued or produced out of the Section, until it meet with the other Side of the Cone continued, viz. *TS* or *Tx*, &c. is call'd the *Transverse Diameter*, or *Transverse Axis* of the *Hyperbola*.



2. All *Right Lines* that are drawn within an *Hyperbola*, at *Right Angles* to its *Axis*, are call'd *Ordinates rightly apply'd*; as in the *Ellipsis* and *Parabola*.

3. That

3. That *Ordinate* which passes thro' the *Focus* of the *Hyperbola* is call'd *Latus Rectum*, or *Right Parameter*, for the same Reason as in the other Sections.

4. The middle Point of the *Transverse Diameter* is call'd the *Center* of the *Hyperbola*: from whence may be drawn *two Right Lines* (out of the Section) call'd *ASYMPTOTES*, because they will always incline (*that is, come nearer and nearer*) to both Sides of the *Hyperbola*, but never meet with (or touch) them, altho' both they and the Sides of the *Hyperbola* were infinitely extended; as will plainly appear in its proper Place.

These five Sections, *viz.* the *Triangle*, *Circle*, *Ellipsis*, *Parabola*, and *Hyperbola*, are all the Plains that can possibly be produced from a Cone; but of them, the *three last* are only called *Conick Sections*, both by the ancient and modern *Geometers*.

Scholium.

Besides the 'foregoing *Definitions*, it may not be amiss to add, by Way of Observation, how one Section may (*or rather doth*) change or degenerate into another.

An *Ellipsis* being that Plain of any Section of the Cone which is between the *Circle* and *Parabola*, 'twill be easy to conceive that there may be great Variety of *Ellipses* produced from the same Cone; and when the Section comes to be exactly parallel to one Side of the Cone, then doth the *Ellipsis* change or degenerate into a *Parabola*. Now a *Parabola*, being that Section whose Plain is always exactly parallel to the Side of the Cone, cannot vary, as the *Ellipsis* may; for so soon as ever it begins to move out of that *Position* (*viz. from being parallel to the Cone's Side*) it degenerates either into an *Ellipsis*, or into an *Hyperbola*: That is, if the Section incline towards the Plain of the Cone's Base, it becomes an *Ellipsis*; but if it incline towards the Cone's Vertex, it becomes an *Hyperbola*, which is the Plain of any Section that falls between the *Parabola* and the *Triangle*. And therefore there may be as many *Varieties* of *Hyperbola's* produced from one and the same Cone, as there may be *Ellipses*.

To be brief, a *Circle* may change into an *Ellipsis*, the *Ellipsis* into a *Parabola*, the *Parabola* into an *Hyperbola*, and the *Hyperbola* into a plain *Isoceles Triangle*: And the Center of the Circle, which is its *Focus* or burning Point, doth, as it were, part or divide itself into two *Foci* so soon as ever the Circle begins to degenerate into an *Ellipsis*; but when the *Ellipsis* changes into a *Parabola*, one End of it flies open, and one of its *Foci* vanishes, and the

the remaining *Focus* goes along with the *Parabola* when it degenerates into an *Hyperbola*: And when the *Hyperbola* degenerates into a plain *Isoceles Triangle*, this *Focus* becomes the *vertical Point* of the *Triangle* (*viz.* the *Vertex* of the *Cone*); so that the *Center* of the *Cone's Base* may be truly said to pass gradually through all the *Sections*, until it arrives at the *Vertex* of the *Cone*, still carrying its *Latus Rectum* along with it: For the *Diameter* of a *Circle* being that *Right Line* which passes through its *Center* or *Focus*, and by which all other *Right Lines* drawn within the *Circle* are regulated and valued, may (*I presume*) be properly called the *Circle's Latus Rectum*: and although it loses the Name of *Diameter* when the *Circle* degenerates into an *Ellipsis*, yet it retains the Name of *Latus Rectum*, with its first Properties, in all the *Sections*, gradually shortening as the *Focus* carries it along from one *Section* to another, until at last it and the *Focus* become co-incident, and terminate in the *Vertex* of the *Cone*.

I have been more particular and fuller in these *Definitions* than is usual in Books of this Subject, which I hope is no Fault, but will prove of Use, especially to a *Learner*: And altho' they may perhaps seem a little strange, and at first hard to be understood, yet, when they are well considered, and compar'd with a *Cone* cut into such *Sections* as have been defined, they will not only be found true, but will also help to form a true and clear *Idea* of each *Section*.

CH A P. II.

Concerning the Chief Properties of an *Ellipsis*.

NOTE, If the transverse *Diameter* of an *Ellipsis*, as *TS* in the following Figure, be intersected or divided into any two *Parts* by an *Ordinate* rightly apply'd, as at the *Points A, C, a, &c.* then are those *Parts TA, TC, Ta, and SA, SC, Sa, &c.* usually called *Abscissæ* (which signifies *Lines* or *Parts* cut off) and by the *Rectangle* of any two *Abscissæ* is meant the *Rectangle* of such two *Parts* as, being added together, will be equal to the *Transverse Diameter*.

$$\text{As } TA + SA = TS. \text{ And } TC + SC = TS.$$

$$\text{Or } TA + SA = TS, \text{ \&c.}$$

See.

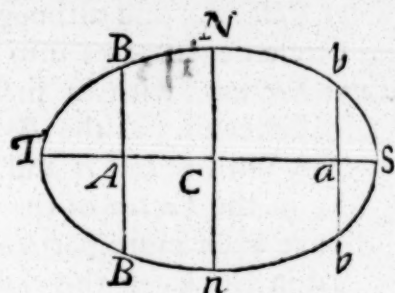
Section 1.

Every *Ellipsis* is proportion'd, and all such Lines as relate to it are regulated, by the Help of one general *Theorem*:

THEOREM. *As the Rectangle of any two Abscissæ : is to the Square of Half the Ordinate which divides them : so is the Rectangle of any other two Abscissæ : to the Square of Half that Ordinate which divides them.*

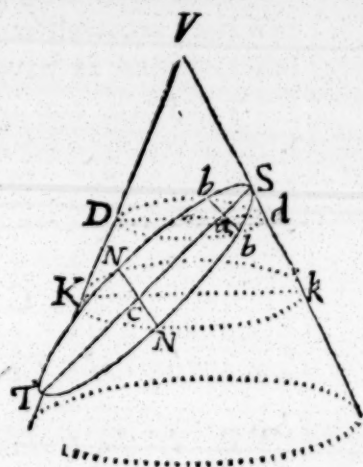
That is,

$$\begin{aligned} TA \times SA : \square BA :: Ta \times Sa : \square ba \\ TA \times SA : \square BA :: TC \times SC : \square NC \\ TC \times SC : \square NC :: Ta \times Sa : \square ba \\ \&c. \end{aligned}$$



Demonstration.

Let the annexed Figure represent a Right Cone, cut thro' both Sides by the Right Line *TS*; then will the Plain of that Section be an *Ellipsis* (by *Seet. 3. Chap. 1.*) *TS* will be the *Transverse Diameter*, *NCN* and *bab* will be *Ordinates rightly apply'd*; as before. Again, if the Lines *Dd* and *Kk* be parallel to the Cone's Base, they will be *Diameters of Circles* (by *Seet. 2. Chap. 1.*) Then will $\triangle TCK$ and $\triangle TaD$ be alike. Also, $\triangle Sad$ and $\triangle SCK$ will be alike.

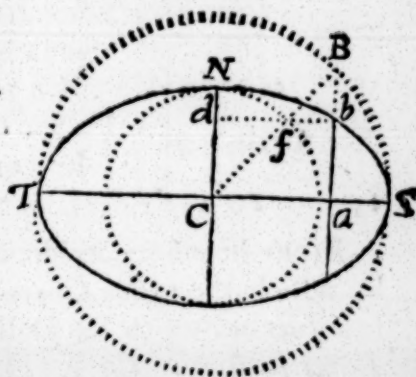


Ergo	1	$Sa : ad :: SC : Ck$	} per Theorem 13.
And	2	$TC : CK :: Ta : aD$	
1	3	$Sa \times Ck = ad \times SC$	
2	4	$Ta \times CK = TC \times aD$	
2 × 3	5	$Sa \times Ck \times Ta \times CK = ad \times SC \times TC \times aD$	Per Axiom 3.
But	6	$CK \times Ck = \square NC$	} per Lemma Seet. 2.
And	7	$aD \times ad = \square ba$	
Then		for $CK \times Ck$, and $aD \times ad$, take $\square NC$ and $\square ba$	
5, 6, 7	8	$Sa \times Ta \times \square NC = TC \times SC \times \square ba$	Per Axiom 5.
Hence	9	$Sa \times Ta : \square ba :: TC \times SC : \square NC$	See Page 194.

Q. E. D.
Or

Or, the Truth of these Proportions may be otherwise prov'd by a Circle, without the Help of the Cone; thus: Let any *Ellipsis* be circumscrib'd and inscrib'd with Circles, as in the following Figure; then from any Point in the circumscrib'd Circle's Periphery, as at *B*, draw the Right Line *Ba*, parallel to the semi-conjugate Diameter *Nc*, then will *ba* be a *Semi-ordinate* rightly apply'd to the transverse Diameter *TS*, as before. Again, from the Point *b* (in the *Ellipsis*'s Periphery) draw the Right Line *bd* parallel to the Transverse *TS*; and draw the Radius *BC*. Then will $\triangle B C a$ and $\triangle C f d$ be alike.

Therefore	1	$\{ \begin{array}{l} B C : B a :: C f : d C \\ \text{per Theorem 13.} \end{array}$
But	2	$\{ \begin{array}{l} T C = B C, N C = C f \\ \text{and } b a = d C \end{array}$
Conseq.	3	$T C : B a :: N C : b a$
Or	4	$T C : N C :: B a : b a$
4 in $\square$'s	5	$\square T C : \square N C :: \square B a : \square b a$
But	6	$\{ \begin{array}{l} T a \times S a = \square B a \\ \text{per Lem. Sect. 2. Ch. I.} \end{array}$
Therefore	7	$\{ \begin{array}{l} T a \times S a : \square b a :: T C \\ \times S C = \square T C : \square N C, \text{ as before.} \end{array}$



And so for any other *Abscissæ* and their *Semi-ordinates*.

These Proportions being found to be the true and common Properties of every *Ellipsis*, all that is farther requir'd in (or about) that *Section* may be easily deduced from them.

Sect. 2. To find the *Latus Rectum*, or *Right Parameter* of any *Ellipsis*.

There are several Ways of finding the *Latus Rectum*, but I think none so easy, and shews it so plainly to be the Third Principal Line in the *Ellipsis*, as the following.

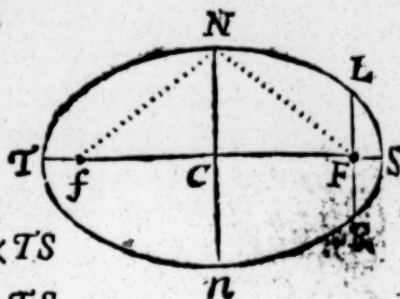
THEOREM. $\{ \begin{array}{l} \text{As the Transverse Diameter : is in Proportion to the} \\ \text{Conjugate : : so is the Conjugate : to the Latus Rectum.} \end{array}$
Viz. (in the following Fig.) $TS : Nn : : Nn : LR$ the *Latus Rectum*.

Demonstration.

From the last Proportions take either of the Antecedents, and its Consequent, viz. either $T C \times S C : \square N C$, or $T a \times S a : \square b a$,
B b b and

and make TS the third Term, to which find a fourth Proportional, and it will be $= LR$:

Thus	1	$TC \times SC : \square NC :: TS : LR$
But	2	$\{ TC = SC$ and $NC = Cn$
Therefore	3	$TC \times SC = \frac{1}{4} \square TS$
And	4	$\square NC = \frac{1}{4} \square Nn$
1, 3, 4	5	$\frac{1}{4} \square TS : \frac{1}{4} \square Nn :: TS : LR$
5	6	$\frac{1}{4} \square TS \times LR = \frac{1}{4} \square Nn \times TS$
6 $\times$ 4	7	$\square TS \times LR = \square Nn \times TS$
7 $\div$ TS	8	$\{ TS \times LR = \square Nn$ which gives the following <i>Analogy</i> .
viz	9	$TS : Nn :: Nn : LR$
Again	10	$\{ TC \times SC : \square NC :: Ta \times Sa : \square ba$ by common <i>Properties</i> .
1, 10	11	$TS : LR :: Ta \times Sa : \square ba.$



From hence 'tis evident that LR , thus found, is that *Ordinate* by which the other *Ordinates* may be regulated and found. Therefore (according to its Definition *Seç. 3, Chap. 1.*) it is the true *Latus Rectum*. Q. E. D.

Conseſtary.

Hence it follows, that if the transverse and conjugate *Diameters* of any *Ellipsis* are given (either in *Lines* or *Numbers*) the *Latus Rectum* may be easily found; and then any *Ordinate*, whose Distance from the Conjugate is given, may be found, as above.

Sect. 3. To find the Focus of any Ellipsis.

The *Focus* is the Distance of the *Latus Rectum* from the Conjugate or Middle of the *Ellipsis* (*vide* Definition 4, Page 364.) and that Distance is always a *Mean Proportional* between the half Sum and half Difference of the transverse and conjugate *Diameters*, which gives this *Theorem*.

THEOREM. $\left\{ \begin{array}{l} \text{From the Square of half the Transverse subtract the} \\ \text{Square of half the Conjugate, the square Root of their} \\ \text{Difference will be the Distance of each Focus from the} \\ \text{Middle or common Center of the Ellipsis.} \end{array} \right.$

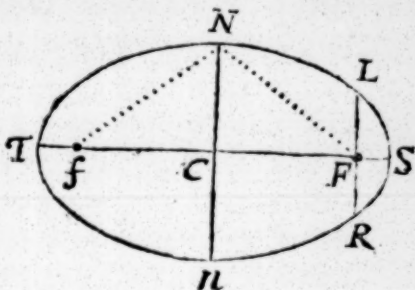
That is, supposing the Points f and F to be the two *Foci*, viz. $fC = Cf$, and $TC = \frac{1}{2} TS$. $NC = \frac{1}{2} Nn$. Then, $TC + NC : fC :: fC : TC - NC$. Ergo $\square FC = TC - \square NC$. Consequently, $FC = \sqrt{\square TC - \square NC}$.

z

Demonſ

Demonstration.

First,	1	$TS \times LR = \square Nn$, by 8th Step of the last Process.
And	2	$TS : LR :: TF \times SF : \square LF$, common Properties.
That is,	3	$TS : LR :: TC + CF \times TC - CF : \frac{1}{4} \square LR = \square LF$
3	4	$\left\{ \begin{array}{l} \frac{1}{4} \square LR \times TS = \\ \square TC - \square CF \times LR \end{array} \right.$
4	5	$\frac{1}{4} LR \times TS = \square TC - \square CF$
5	6	$\frac{1}{4} TS \times LR = \frac{1}{4} \square Nn = \square NC$
6	7	$\square NC = \square TC - \square CF$
7	8	$\square CF = \square TC - \square NC$
8	9	$CF = \sqrt{\square TC - \square NC}$



Now from hence is deduced that notable Proposition, upon which is grounded the usual Method of describing an *Ellipsis*, and drawing of *Tangents*, &c.

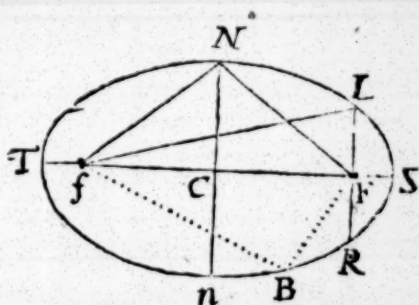
PROPOSITION.

If from the two Foci of any Ellipsis there be drawn two Right Lines, so as to meet each other in any Point of the Ellipsis's Periphery, the Sum of those Lines will be equal to the Transverse.

Viz. $fN \times NF = TS$. $fL \times LF = TS$. Or $fB + BF = TS$, &c.

Demonstration.

First,	1	$\left\{ \begin{array}{l} \square CF \times \square NC = \square TC \\ \text{by 8th of the last.} \end{array} \right.$
But	2	$\left\{ \begin{array}{l} \square CF + \square NC = \square NF \\ \text{by Theorem II.} \end{array} \right.$
1,	3	$\left\{ \begin{array}{l} \square NF = \square TC \\ \text{by Axiom 5.} \end{array} \right.$
3 nd	4	$NF = TC$
Hence	5	$2 NF = 2 TC = TS$
Again,	6	$TS : LR :: TF \times FS : \square LF$, by common Properties.
Conseq.	7	$\frac{1}{2} TS : \frac{1}{2} LR :: TF \times FS : \square LF$
But,	8	$\frac{1}{2} TS = TC$. And $\frac{1}{2} LR = LF$
Ergo	9	$TC : LF :: TC + CF \times TC - CF : \square LF$
7	10	$TC \times LF = \square TC - \square CF$
But,	11	$\square fF + \square LF = \square fL$, by Theorem II.
That is,	12	$4 \square CF + \square LF = \square fL$, for $2 CF = fF$
8 $\times$ 4	13	$4 \square TC - 4 \square CF = 4 TC \times LF$
10 + 11	14	$4 \square TC + \square LF = 4 TC \times LF + \square fL$
12 -	15	$4 \square TC - 4 TC \times LF + \square LF = \square fL$



$$\begin{array}{l|l}
 13 \quad \omega^2 & 14 \quad 2 TC - LF = fL \\
 14 + LF & 15 \quad 2 TC = fL + LF. \quad \text{But } 2 TC = TS \\
 \text{Ergo} & TS = fL + LF.
 \end{array}
 \quad \text{Q. E. D.}$$

And this *Proposition* must needs hold true to every Point in the *Ellipsis's* Periphery, viz. at *B*, &c. As will evidently appear to any one who rightly considers, That, as a Thread just the Length of the Diameter of any Circle having its two Ends ty'd together, and then mov'd about a Point in the Center (viz. by making it a double Radius) will, by drawing another Point in its Extremity, describe the Periphery of a Circle; [*vide* Definition Page 280] even so, if a Thread just the Length of the transverse Diameter (*TS*) having its two Ends so fix'd upon the two Foci (*f* and *F*) that it may be mov'd about them, by drawing a Point in its Extremity (viz. at its full Stretch) it will describe the true Periphery of an *Ellipsis*.

Now, altho' this easy Way of describing, or, as usually phras'd, drawing an *Ellipsis*, be mechanical, and known even to most Joiners, Carpenters, &c. yet it gives as compleat and clear an Idea of that Figure as any other Way whatsoever; and by describing it thus about its two Foci, as a Circle is about its Center, doth plainly shew that those two Points are not improperly call'd particular Centers in Definition 4, Sect. 3, Chap. 1. for each of them bears much the same Respect to the *Ellipsis's* Periphery, as the Circle's Center doth to its Periphery.

Sect. 4. To describe or delineate an *Ellipsis* several Ways.

There are several (*other*) Ways of describing an *Ellipsis*, both Geometrically and Numerically, according to peculiar Occasions, but I shall only mention two or three of them, leaving the rest to the Learner's Genius. Now, in order to that Work, it will be convenient to consider what Lines are requisite to limit or bound its Form, which I take to be chiefly these following.

1. If the Transverse and Conjugate are given, the *Ellipsis* is perfectly limited; (*vide* Consecutary Page 363.) for if *TS* and *Nn* be set at Right Angles in their Middle at *C*, and *TC* or *CS* be set off from *N*, or *n*, both Ways upon the Transverse to *f* and *F*, (viz. make $fN = TC = NF$) then will those Points *f* and *F* be the two Foci (by 4th Step of the last Process) and then the *Ellipsis* may be describ'd as above.

2. If

2. If the *Transverse Diameter* and *Latus Rectum* are given, the *Ellipsis* is truly limited, because by them the *Conjugate* may be found, by Sect. 2.

3. Or if only the *Transverse*, and the Proportion it hath either to the *Conjugate* or *Latus Rectum*, be given, the *Ellipsis* is thereby limited. As for Instance; suppose the given *Ratio* between the *Transverse* and *Conjugate* to be, as $a : d$:

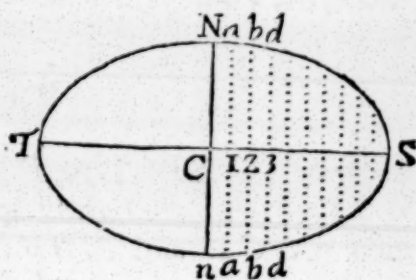
Viz. $a : d :: TS : Nn$, then $\frac{TS \times d}{a} = Nn$, &c.

4. If either the *Transverse* or *Conjugate*, and the Distance of the *Focus* from the *Conjugate* be given, the *Ellipsis* is limited, because by them the *Conjugate* or *Transverse* may be found.

These being premis'd, and the *precedent Work* a little consider'd, it must be easy to describe or delineate any *Ellipsis* in *Plano*, either Geometrically or Numerically.

1. To describe an Ellipsis Numerically by Points.

Suppose the *Transverse Diameter* $TS = 20$, and the *Conjugate* $Nn = 12$, (either Inches, or any other equal Parts) and let them cross each other at *Right Angles* in their Middles, as in the Point C ; then will $TC = CS = 10$, and $NC = Cn = 6$, and it will be $20 : 12 :: 12 : 7$, 2 = the *Latus Rectum*.



Again $20 : 7$, 2. Or rather take their *Ratio*.

Thus $\begin{cases} 1 : 0, 36 :: \frac{10+1}{10-1} \times \frac{10-1}{10-1} : \square a. \parallel 1. \\ 1 : 0, 36 :: \frac{10+2}{10-2} \times \frac{10-2}{10-2} : \square b. \parallel 2. \\ 1 : 0, 36 :: \frac{10+3}{10-3} \times \frac{10-3}{10-3} : \square d. \parallel 3. \text{ \&c.} \end{cases}$

Viz. $\begin{cases} \frac{100-1}{100-4} \times 0,36 = \square a.1. \text{ Hence } \sqrt{90 \times 0} = 5,97 \text{ \&c.} = a. 1 \\ \frac{100-4}{100-9} \times 0,36 = \square b.2. \quad \sqrt{90 \times 0} = 5,88 \text{ \&c.} = b. 2 \\ \frac{100-9}{100-9} \times 0,36 = \square d.3. \quad \sqrt{91 \times 0,36} = 5,72 \text{ \&c.} = d. 3 \end{cases}$

If so many *Semi-ordinates* as may be thought convenient (*the more the better*) be found in this Manner, and every one of them be set off at *Right Angles* from its respective Point in the *Transverse Diameter* each Way, viz. from 1 to a , from 2 to b , from 3 to d , &c. Then if a Curve Line be carefully drawn with an even Hand thro' those extreme Points a, b, d , &c. it will be the *Ellipsis's Periphery* requir'd.

2. To describe an Ellipsis Geometrically by Points.

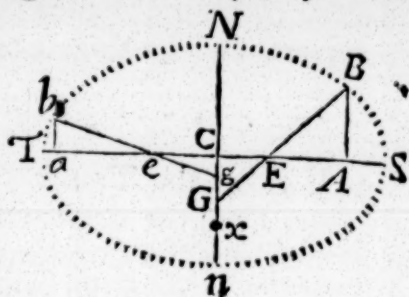
Having the Transverse and Conjugate Diameters given, viz. TS and Nn , placed at *Right Angles* in their *Middles*, as before: Then from either End of the Conjugate, viz. N (or n) set off half the Transverse Diameter to x .

That is, make $Nx = TC$ (continuing the Conjugate Nn when it is shorter than TC) Or, which is all one, make $Cx = TC - NC$.

Then take any Point in the Line Cx at Pleasure; suppose it at G , and from that Point at G set off the Distance Cx to the Transverse

(as at E) viz. make $GE = Cx$, and join the Points $G E$ with a Right Line, produced so far beyond E as to make $EB = NC$. Consequently $GB = TC$.

Then, I say, where-ever the Point G was taken between C and x , the Point B will just touch (or fall in) the Ellipsis's Periphery.



Demonstration.

Draw the Right Line BA perpendicular to TS , viz. let BA be a *Semi-ordinate* rightly apply'd to the transverse Diameter TS ; then $\triangle GCE$ and $\triangle BAE$ will be alike.

Consequently	1	$CE : AE :: EG : EB$, by Theorem 13.
1, and	2	$CE + AE : AE :: EG + EB : EB$. See p. 192.
But	3	$CE + AE = CA$. $EG + EB = TC$. And $EB = NC$
Therefore	4	$CA : AE :: TC : NC$
6, in $\square$'s	5	$\square CA : \square AE :: \square TC : \square NC$
5, $\therefore$	6	$\frac{\square CA \times \square NC}{\square TC} = \square AE$
But	7	$\square NC - \square AB = \square AE$
That is,		$\square EB - \square AB = \square AE$
6, 7	8	$\frac{\square CA \times \square NC}{\square TC} = \square NC - \square AB$
8 $\times$ $\square TS$	9	$\square CA \times \square NC = \square NC \times \square TC - \square AB \times \square TC$
9 $\div$	10	$\square NC \times \square TC - \square CA \times \square NC = \square AB \times \square TC$
10, Analogy	11	$\square TC : \square NC :: \square TC - \square CA : \square AB$
That is,	12	$TC \times CS : \square NC :: TC + CA \times TC - CA : \square AB$

which is according to the common *Properties* of the *Ellipsis*; Therefore the Point B is truly found. Q. E. D.

Hence

Hence it follows, that if a convenient Number of such Lines as $GE B$ be so drawn (as above directed) from the like Number of Points taken between C and x , &c. their extream Points (as at B) will be those Points by which (*with an even Hand*) the *Ellipsis* may be truly describ'd, as before.

But, if this be well understood, it will be very easy to conceive how to describe an *Ellipsis* very readily, without drawing those Lines, by having a thin, streight, narrow *Ruler* just the Length of TC , made somewhat sharp at both Ends, upon which, from one of its Ends, set off the Length of NC . Then, if the Point upon the *Ruler* which represents E be gradually or easily moved along the Transverse TS , and at the same Time the Point or End representing G be kept sliding close along the Conjugate Nn , 'tis evident from the Work above, that the End of the *Ruler* representing B will, by that Motion, assign the true *Periphery* of the *Ellipsis* required; for by that Motion the streight Edge of the *Ruler* doth supply an infinite Number of the aforesaid Lines; as will appear very plain and easy in Practice.

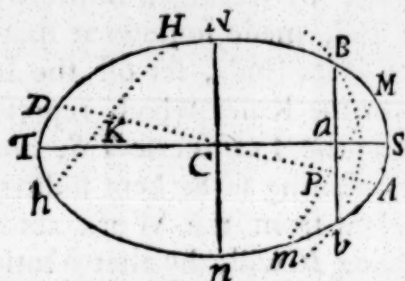
Scholium.

Now from hence was deduced the first Invention of that well-contrived *Instrument* for drawing an *Ellipsis* by one Motion, commonly called the *Elliptical Compasses*, being usually made of Brass, and compos'd of three Parts, two of which represent (*or rather supply*) the transverse and conjugate Diameters set together at *Right Angles*; and the third Part is a moveable *Ruler*, which performs the Office of the last-mentioned thin *Ruler*. But because the making of it is so well known to most Mathematical Instrument-makers, especially to that accurate and ingenious Artist Mr JOHN ROWLEY, Mathematical Instrument-maker, under St. Dunstan's Church in Fleet-street, London; who, for his great Skill in contriving, framing, and graduating all kind of Mathematical Instruments, may, I believe, be justly called one of the best Workmen of his Trade in Europe; I think it needless therefore to give a particular Description of that *Instrument*.

Also from hence came that ingenious Invention of making Engines for turning all Sorts of elliptical or oval Work, as oval Boxes, Picture-Frames, &c.

Sect. 5. *Any Ellipsis being given, to find its Transverse and Conjugate Diameters.*

Suppose the given *Ellipsis* to be $TNSn$ (in the annexed Scheme) in which let it be required to find the transverse Diameter TS and its Conjugate Nn . Draw within the *Ellipsis* any two *Right Lines* parallel to each other as Hb and Mm , and bisect those Lines, viz. find the Middle Point of each, as at K and P ; then thro' those Points K and P draw a *Right Line*, as DA , and it will be a Diameter; for it will divide the *Ellipsis* into two equal Parts, [See Defn. 1, Page 363.] consequently the Middle of DA will be the true Middle or common Center of the *Ellipsis*, as at C .



For 'tis the Nature and Property of all Diameters, howsoever they are drawn in any *Ellipsis* (as 'tis in a Circle) to cut or cross one another in the common Center or Middle of the Figure, as at C .

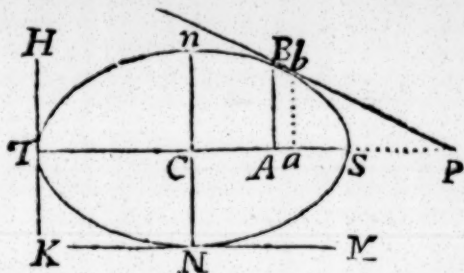
Upon the Point C describe an *Arch* of any Circle that will cut the *Ellipsis's* Periphery in two Points, as at B and b ; then join those Points Bb with a *Right Line*, and it will be an *Ordinate*, thro' whose Middle (as at a) and the common Center C , the transverse Diameter TS must pass. For $BS = Sb$, and Ba is at Right Angles with TS ; therefore the Line Bb is an *Ordinate* rightly apply'd to TS the transverse Diameter. And if thro' the Point C there be drawn the *Right Line* Nn parallel to Bb , it will become the Conjugate; as was requir'd.

Sect. 6. *To draw a Tangent, or Right Line that may touch the Ellipsis's Periphery in any assigned Point.*

The Drawing of *Tangents* to or from any assigned Point in the *Ellipsis's* Periphery, admits of three Cases.

Case 1. If it be requir'd to draw a *Tangent* that may touch the *Ellipsis* in either of the extream Points of its transverse Diameter, as at T or S , it is plain the *Tangent* must be drawn parallel to the conjugate Diameter Nn ; as HK in the following Figure is suppos'd to be.

Case 2. Or, if the *Tangent* must be drawn to touch the *Ellipsis* in either of the extream Points of its *Conjugate Diameter*, as at *N* or *n*, 'tis as evident that it must be drawn parallel to the *Transverse Diameter* *TS*, as *KM*. Consequently if that *Tangent* and the *Transverse* were both infinitely continu'd, they would never meet.



Case 3. But if it be requir'd to draw a *Tangent* that may touch the *Ellipsis* in any other Point, as at *B*, &c. Then, if the *Tangent* and the *Transverse Diameter* be both continu'd, they will meet in some Point, as at *P*; and those two Points (viz. *B* and *P*) do so mutually depend upon each other, that one of them must be assigned in order to find the other, that so the *Tangent* may by them be truly drawn. Let $D = TS$, $y = AS$, and $z = AP$. Then, if y be given, z may be found by this

Theorem $\left\{ \frac{Dy - yy}{\frac{1}{2}D - y} = z. \right.$ Or, if z be given, y may be found by this *Theorem* $\left\{ \frac{D + z}{2} \pm \sqrt{\frac{DD + zz}{4}} = y. \right.$

Demonstration.

Draw the *Semi-ordinate* *ba*, as in the *Figure*; then will $\triangle BAP$ and $\triangle baP$ be alike. Put $x = Aa$ the *Distance* between the two *Semi-ordinates* (viz. between *BA* and *ba*) which we suppose infinitely small.

Then	1	$z : z - x :: BA : ba$, by <i>Theorem 13</i> .
But	2	$D - y \times y : D - y + x \times y - x :: \square BA : \square ba$
That is,	3	$Dy - yy : Dy - yy + 2bx - Dx - xx :: \square BA : \square ba$
1 in $\square$'s	4	$zz : zz - 2zx + xx :: \square BA : \square ba$
Suppose	5	$x = c$, that so x may be every where rejected.
3, Then	6	$Dy - yy : Dy - yy + 2y - D :: \square BA : \square ba$
4, And	7	$zz : zz - 2z :: \square BA : \square ba$
6, 7	8	$Dy - yy : Dy - yy + 2y - D :: zz : zz - 2z$
8	9	$2yzz - Dzz = 2yyz - 2Dyz$
9 $\div 2z$	10	$yx - \frac{1}{2}Dz = yy - Dy$
10 $\pm$	11	$\frac{1}{2}Dz - yz = Dy - yy$

C c c

11 $\div$

Case

11 ÷	12	$z = \frac{Dy - yy}{\frac{1}{2}D - y}$	} which is the 1st Theorem, and gives the following Analogy.
Analogy	13	$\frac{1}{2}D - y : y :: D - y : z.$	
10 — yz	14	$yy - Dy - yz = -\frac{1}{2}Dz$	
14 C □	15	$yy - Dy - yz + \frac{1}{4}DD - \frac{1}{2}Dz + \frac{1}{4}zz = \frac{1}{4}DD + \frac{1}{4}zz$	
15 w ²	16	$y - \frac{1}{2}D - \frac{1}{2}z = \sqrt{\frac{1}{4}DD - \frac{1}{4}zz}$	
That is,	17	$y = \frac{1}{2}D + \frac{1}{2}z \pm \sqrt{\frac{1}{4}DD + \frac{1}{4}zz}$	} which is the 2d Theor. Q. E. D.

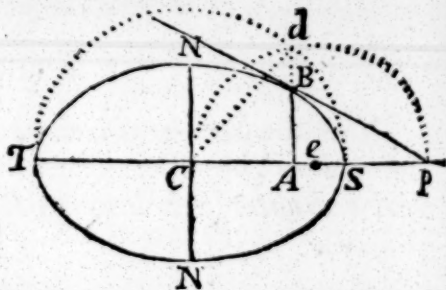
The Geometrical Performance of these two Theorems is very easy, as by the following Figure.

1. Suppose the Point *B* in the Ellipsis Periphery were given, and it were requir'd to find the Point *P*, &c.

Make *TC* Radius, and upon the common Center *C* describe the Semicircle *TdS*, and join the Points *C* and *d* with a Right Line; then bisection that Line (by Prob. 2, Page 287) and mark the Point where the bisecting Line would cross the Transverse, as at *e*. Upon that Point *e*, with the Radius *Ce* (or *Cd*) describe another Semicircle, producing the Transverse Diameter to its Periphery, and it will assign the Point *P*.

For if $D = TS$, $y = AS$, $z = AP$, as before.

Then	1	$D - y \times y = \square dA$
And	2	$\frac{1}{2}D - y \times z = \square dA$
For	3	$TA : dA :: dA : SA$
And	4	$CA : dA :: dA : AP$
But	5	$CA = \frac{1}{2}D - y$, &c.
1, 2	6	$\left\{ \begin{array}{l} \frac{1}{2}Dz - yz = Dy - yy \\ \text{as at the 11th Step before} \end{array} \right.$



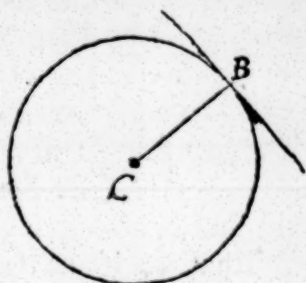
Therefore the Point *P* is truly found. Consequently, if a Right Line be drawn through those Points *B* and *P*, it will be the Tangent requir'd, according to the first Theorem.

2. The Converse of this is as easy, to wit, if the Point *P* be given, thence to find the Point *B* in the Ellipsis Periphery. Thus, circumscribe half the Ellipsis with the Semicircle *TdS*, as before; and bisection the Distance between the Points *C* and *P*, as at *e*, viz. Let $Ce = eP$. Then making *Ce* Radius, upon the Point *e*, describe the Semicircle *CdP*; and from the Point where the two Semicircles intersect or cross each other, as at *d*, draw the Right Line *dA* perpendicular to the Transverse *TS*, and it will assign the

the Point of Contact *B* in the *Ellipsis Periphery* through which the *Tangent* must pass.

But the *Practical Method* of drawing *Tangents* to any assign'd Point in the *Ellipsis Periphery* may (without finding the *aforesaid Point P*) be easily deduced from the following *Property of Tangents* drawn to a *Circle*, which is this:

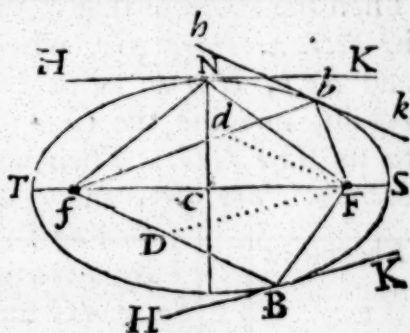
If to any *Radius* of a *Circle*, as *CB*, there be drawn a *Tangent Line* (as *HK*) to touch the *Radius* at the *Point B*; the two *Angles*, which the *Tangent* makes with the *Radius*, will always be two *Right Angles* (16, 17, 18, 19 *Euclid* 3.) that is, $\angle HBC = \angle CBK = 90^\circ$.



In like Manner the two *Angles*, made between the *Tangent* and the two Lines drawn from the *Foci* of any *Ellipsis* to the *Point of Contact*, will always be equal, but not *Right Angles*, save only at the two Ends of the *Transverse Diameter*.

These being well consider'd, and compar'd with what hath been said in *Page 366*, it must needs be easy to understand the following *Way of drawing Tangents* to any assign'd *Point* in the *Ellipsis Periphery*; which is thus:

Having by the *transverse* and *conjugate Diameters* found the two *Foci* *f* and *F*, by *Seēt. 3.* from them draw two *Right Lines* to meet each other in the assign'd *Point of Contact*, as *fb* and *Fb* (or *fB* and *FB*) in the annex'd *Figure*. Next set off (viz. make) $bd = bF$ (or $BD = BF$) and join the *Points Fd* (or *FD*) with a *Right Line*.



Then, I say, if a *Right Line* be drawn through the *Point of Contact* *b* (or *B*) parallel to *dF*, or *DF*, it will be the *Tangent* requir'd. For it is plain, that as the $\angle fNH = \angle FNK$ when the *Tangent* is parallel to the *Transverse Diameter*, even so is the $\angle fbb = \angle FBk$, (and $\angle fBH = \angle FBK$) and will be every where so, as the *Point of Contact* *b* (or *B*) and its *Tangent* is carried about the *Ellipsis Periphery* with the *Lines fbF* (or *fBF*).

C H A P. III.

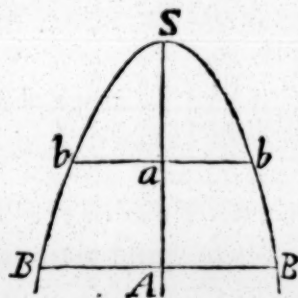
Concerning the Chief Properties of every Parabola.

NOTE, in every Parabola, the intercepted Diameter, or that Part of its Axis, which is between the Vertex and that Ordinate which limits its Length, as Sa or SA , &c. is call'd an *Abscissa*. *Sect.* 1. The Plain or Figure of every Parabola is proportion'd by its Ordinates and *Abscissæ*, as in the following Theorem:

THEOREM $\left\{ \begin{array}{l} \text{As any one } Abscissa : \text{ is to the Square of its Semi-} \\ \text{ordinate} :: \text{ so is any other } Abscissa : \text{ to the Square of} \\ \text{its Semi-ordinate.} \end{array} \right.$

That is, if we suppose the annex'd Figure to be a Parabola, wherein Sa , and SA , are *Abscissæ*, and ba , BAB , Ordinates rightly apply'd, it will

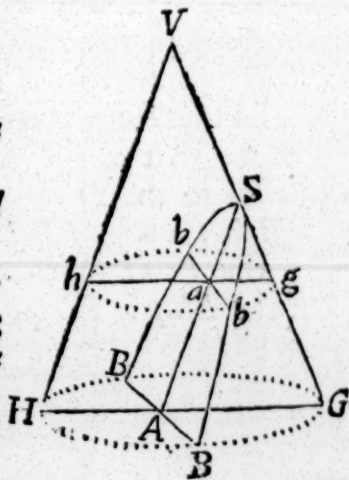
be $Sa : \square ba :: SA : \square BA$ $\left\{ \begin{array}{l} \text{wheresoever} \\ \text{the Points } a, \\ A, \text{ are taken,} \end{array} \right.$
or $Sa : SA :: \square ba : \square BA$
And so for any other *Abscissæ*, &c.



Demonstration.

Let the following Figure HVG represent a Right Cone cut into two Parts by the Right Line SA , parallel to its Side VH . Then the Plain of that Section, viz. $BbSbB$, will be a Parabola, by *Sect.* 4. Page 364. wherein let us suppose SA to be its Axis, and ba , BAB to be Ordinates rightly apply'd to that Axis. Again, imagine the Cone to be cut by the Right Line hg parallel to its Base HG . Then will hg be the Diameter of a Circle, by *Sect.* 2. Page 363. and $\triangle Sag$ like to $\triangle SAG$.

Therefore	1	$\left\{ \begin{array}{l} Sa : ag :: SA : AG \\ \text{By Theorem 13.} \end{array} \right.$
I	2	$Sa \times AG = SA \times ag$
$2 \times ba$	3	$\left\{ \begin{array}{l} Sa \times AG \times ba = SA \times ag \times ba \\ \text{By Axiom 3.} \end{array} \right.$
But	4	$\left\{ \begin{array}{l} HA = ba, \text{ because } SA \\ \text{is parallel to } VH \end{array} \right.$
And	5	$\left\{ \begin{array}{l} \square BA = AG \times HA \text{ } \{ \text{By Lem.} \\ \square ba = ag \times ba \text{ } \{ \text{P. 363} \end{array} \right.$
3, 4, 5,	6	$\left\{ \begin{array}{l} Sa \times \square BA = SA \times \square ba \\ \text{By Axiom 5.} \end{array} \right.$
6, Analogy	7	$\left\{ \begin{array}{l} Sa : \square ba :: SA : \square BA \\ \text{Vide Page 194.} \end{array} \right.$



Q. E. D.

These

These Proportions being prov'd to be the common Property of every Parabola, all that is farther requir'd about that Section, or Figure, may from thence easily be deduced.

Sect. 2. To find the Latus Rectum or Right Parameter of any Parabola.

The Latus Rectum of a Parabola hath the same Ratio or Proportion to any Abscissa, and its Semi-Ordinate, as the Latus Rectum of any Ellipsis hath to its Transverse and Conjugate Diameters, and may be found by this Theorem.

THEOREM $\left\{ \begin{array}{l} \text{As any Abscissa : is in Proportion to its Semi-ordinate} \\ \text{: : so is that Semi-ordinate : to the Latus Rectum.} \end{array} \right.$

Let $L =$ the Latus Rectum.

Then	1	$Sa : ba :: ba : L$	$\left\{ \begin{array}{l} \text{where-ever the Points } a, \text{ and} \\ A, \text{ are taken in the Axis.} \end{array} \right.$
And	2	$SA : BA :: BA : L$	
1	3	$\frac{\square ba}{Sa} = L : \text{Or } Sa \times L = \square ba$	
2	4	$\frac{\square BA}{Sa} = L : \text{Or } SA \times L = \square BA$	
3 = 4	5	$\frac{\square BA}{SA} = \frac{\square Sa}{ba}$ Per Axiom 5.	
5 $\times$	6	$Sa \times \square BA = SA \times \square ba$, which gives this	
Analogy	7	$Sa : \square ba :: SA : \square BA$, the same as at the 7th	

Step of the last Process; therefore L (thus found) is the true Latus Rectum, by which all the Ordinates may be regulated and found, according to its Definition in Section 4, Page 364. For by the third Step $Sa \times L = \square ba$, and by the 4th Step $SA \times L = \square BA$. Consequently $\sqrt{Sa \times L} = ba$ and $\sqrt{SA \times L} = BA$; and so for any other Ordinate.

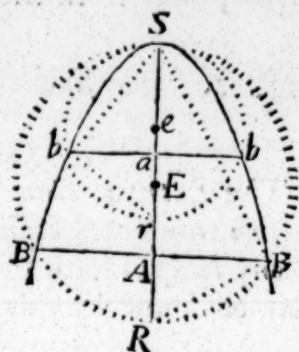
Or if the Ordinates are given, to find their Abscissæ; then it will be, $L : ba :: ba : Sa$, and $L : BA :: BA : SA$, &c.

Consequently $\frac{\square ba}{L} = Sa$, and $\frac{\square BA}{L} = SA$, &c.

From the Consideration of these Proportions, it will be easy to conceive how to find the Latus Rectum Geometrically, thus :

Join

Join the vertical Point S of the Axis, and either extream Point of any Ordinate as B (or b) with a Right Line, viz. SB (or Sb) and bisect that Line (by Problem 2. Page 287.) marking the Point where the bisecting Line doth intersect or cross the Axis, as at E (or e) and with the Radius SE (or Se) upon the Point E (or e) describe a Circle; (as in the annex'd Figure) then will the Distance between the Ordinate and that Point where the Circle's Periphery cuts the Axis, viz. AR (or ar) be the true Latus Rectum required.



For $SA : BA :: BA : AR$, and $Sa : ba :: ba : or$, by Theor. 13. therefore $AR = L$. And $ar = L$, by the 1st and 2d Steps above.

Confectary.

From these Proportions of finding the Latus Rectum, it will be easy to deduce and demonstrate the following Theorem:

THEOREM { As the Latus Rectum : is to the Sum of any two Semi-ordinates : : so is the Difference of those two Semi-ordinates : to the Difference of their Abscissæ.

Suppose any Right Line drawn within the Parabola, as bD , parallel to its Axis SA ; then will that Line (viz. bD) be a Diameter (by Def. 5, Page 365) which will make $ED = AB + ab$, $DB = AB - ab$, and $bD = SA - Sa$. Then it will be $L : ED :: DB : bD$, according to the Theorem.

Demonstration.

First	1	{	$SA = \frac{\square BA}{L}$, by Step 2.	
			of the last Procefs.	
And	2	{	$Sa = \frac{\square ba}{L}$ by Step 1.	
			of the last Procefs.	
1 — 2	3		$SA - Sa = \frac{\square BA - \square ba}{L}$	
3 × L	4		$SA - Sa \times L = \square BA - \square ba$	{ Which gives the following Analogy.
But	5		$\square BA - \square ba = BA + ba \times BA - ba$	
4, = 5, 6	6		$SA - Sa \times L = BA + ba \times BA - ba$	
6, Analogy	7		$L : BA + ba :: BA - ba : SA - Sa$	
Or	8		$L : ED :: DB : bD$	This

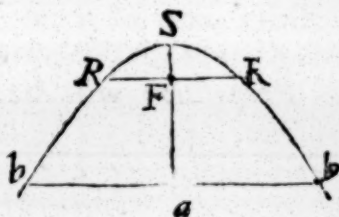
This peculiar Property of the Parabola was first publish'd, Anno 1684, by one Mr. Thomas Baker, Rector of Bishop Nympton in Devonshire, in a Treatise entitled, *The Geometrical Key: Or, the Gate of Æquations unlock'd*; wherein he hath shew'd the Geometrical Construction and Solution of all Cubick and Biquadratick Adfectèd Æquations by one general Method, which he calls a *Central Rule*, deduced from this pecculiar Property of the Parabola.

Sect. 3. To find the Focus of any Parabola.

The Focus of every Parabola is that Point in its Axis through which the Latus Rectum doth pass. (See Definition 3. Sect. 4. Page 359.) Therefore its Distance from the Vertex of the Parabola may be easily found, either by the Latus Rectum itself, or by any other Ordinate, and its Abscissæ.

Thus, suppose the Point at F to be the Focus, S the Vertex, the Ordinate RFR = L the Latus Rectum, and b a b any other Ordinate. Then will $SF = \frac{1}{2} L$.

Or $SF = \frac{\square ba}{4sa}$



Demonstration.

First	1	$SF \times L = \square FR$. by Sect. 2. Page 375.
And	2	$FR = \frac{1}{2} L$; for the Ordinate RFR = L as above.
2 $\odot^2$	3	$\square FR = \frac{1}{4} \square L = \frac{1}{2} L \times \frac{1}{2} L$
1, = 3	4	$SF \times L = \frac{1}{4} \square L$
4 $\div L$	5	$SF = \frac{1}{4} L$, as by Definition 4. Sect. 4. Page 359.
Again	6	$\frac{\square ba}{sa} = L$, by the third Step in Page 375.
Conseq.	7	$\frac{\square ba}{4sa} = \frac{1}{4} L$, &c. as above. Q. E. D.

Sect. 4. To describe, or draw a Parabola several Ways.

Note, There are two or three Ways of drawing a Parabola instrumentally at one Motion; but because those Instruments or Machines are not only too perplex'd for a Learner to manage, but also a little subject to Error, I have therefore chosen to shew how that Figure may be (the best) drawn by a convenient Number of Points, viz. Ordinates found, either Numerically or Geometrically, according to the DATA; which if the Work of the three last Sections be well consider'd, must needs be very easy.

1. If any *Ordinate* and its *Abscissa* are given, there may by them be found as many *Ordinates* as you please to assign or take *Points* in the *Parabola's Axis*; (by *Seet. 4. Page 380*) and the *Curve* of the *Parabola* may be drawn by the extream *Points* of those *Ordinates*, as the *Ellipsis* was *Page 373*.

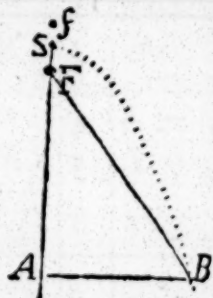
2. If the *Latus Rectum*, and either any *Ordinate*, or its *Abscissa*, are given, then any assign'd Number of *Ordinates* may by them be found (by *Seet. 2. Page 381*) either *Numerically* or *Geometrically*, &c.

3. If only the Distance of the *Focus* from the *Vertex* of the *Parabola* be given, any assign'd Number of *Ordinates* may be found by it. For $SF = \frac{1}{4}L$ the *Latus Rectum*, and $\frac{1}{2}L = FR$ as in the last *Section*; and it will be, as SF : is to $\square FR$:: so is any other *Abscissa*, viz. (Sa or SA , &c.) : to the *Square* of its *Semi-ordinate* (viz. $\square ba$, or $\square BA$) according to the common *Property* of the *Parabola*.

Altho' any of these *Ways* of finding the *Ordinates* are easy enough, yet that *Way* which may be deduced from the following *Proposition* will be found much more easy, and ready in *Practice*.

PROPOSITION. $\left\{ \begin{array}{l} \text{The Sum of any } Abscissa \text{ and focal Distance from} \\ \text{the Vertex, will be equal to the Distance from} \\ \text{the Focus to the extream Point of the Ordinate,} \\ \text{which cuts off that } Abscissa. \end{array} \right.$

For Instance, suppose S to be the *Vertex* of any *Parabola*, the Point F to be its *Focus*, and AB any *Semi-ordinate* rightly apply'd to its *Axis SA*: Then, I say, where-ever the Point A is taken in the *Axis*, it will be $SA + SF = FB$. Consequently, if $Sf = SF$, it will be $fA = FB$.



Demonstration.

First	1	$SF = \frac{1}{4}L$ by the 7th Step, <i>Seet. 3</i> .
Ergo	2	$fA = FA + \frac{1}{2}L$ by <i>Construction</i> above.
2 $\times$	3	$\square fA = \square FA + FA \times L + \frac{1}{4}LL$
Again	4	$SA = FA + \frac{1}{4}L$ by the <i>Supposition</i> and <i>Figure</i> .
4 $\times L$	5	$SA \times L = FA \times L + \frac{1}{4}LL$, but $SA \times L = \square AB$
Ergo	6	$\square AB = FA \times L + \frac{1}{4}LL$
3 — 6	7	$\square fA - \square AB = \square FA$, conse. $\square fA = \square FA + \square AB$
But	8	$\square FA + \square AB = \square FB$, by <i>Theorem II</i> .
Ergo	9	$\square fa = \square FB$
9 $\sqrt{}$	10	$fA = FB$

Q. E. D.

This

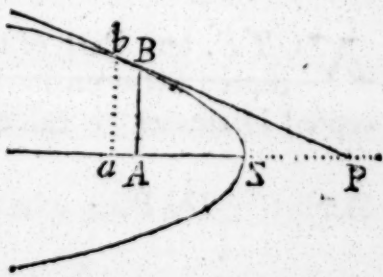
This Proposition being well understood, 'twill be very easily apply'd to *Prælice*, supposing the *Focal Distance* given, or any other *Data* by which it may be found. Thus draw any *Right Line* to represent the *Parabola's Axis*, and from its *vertical Point*, as at *S*, set off the *Focal Distance* both upwards and downwards, viz. make $Sf = SF$, the Distance of the given *Focus* from the *Vertex*; as in the *Scheme*: Then by the Proposition 'tis evident, that, if never so many Lines be drawn *Ordinately* at *Right Angles* to the *Axis*, the true Distance between the Point *f* out of the *Parabola*, and any of those Lines (or *Ordinates*) being measur'd or set off from the *Focus F* to the same Line or Ordinate, 'twill assign the true Point in that Line through which the *Curve* must pass; that is, it will shew the true Limits or Length of that *Ordinate*; as at *B* in the last Scheme.

Proceeding on in the very same Manner from *Ordinate* to *Ordinate*, you may with great Expedition and Exactness find as many *Ordinates* (or rather their Points only, like *B*) as may be thought convenient, which, being all join'd together with an even Hand, will form the *Parabola* requir'd.

N. B. The more *Ordinates* (or their Points) there are found, and the nearer they are to one another, the easier and exacter may the *Curve* of the *Parabola* be drawn. The same is to be observ'd when any other *Curve* is requir'd to be drawn by Points.

Sect. 5. To draw a *Tangent* to any given Point in the *Curve* of a *Parabola*.

Tangents are very easily drawn to the *Curve* of any *Parabola*; For, supposing *S* to be its *Vertex*, *B* the Point of Contact (viz. the Point where the *Tangent* must touch the *Curve*) and *P* the Point where the *Tangent* will intersect (or meet with) the *Parabola's Axis* produced: Then if from the Point of Contact *B* there be drawn the *Semi-ordinate* *BA* at Right Angles to the *Axis* *SA*, wheresoever the Point *A* falls in the *Axis*, 'twill be $SP = SA$.



Demonstration.

Draw the *Semi-ordinate* *ba* (as in the Figure) then will the $\triangle BAP$ and $\triangle b a P$ be alike. Let $y = AS$ the *Abscissa*, and $z =$
 $\quad \quad \quad D d d \quad \quad \quad SP;$

SP; put $x = Aa$ the Distance between the two *Semiordinates*, which we suppose to be infinitely near each other, as in the *Ellipsis*, Page 377.

Then	1	$y + z : BA :: y + z + x : ba$, per <i>Theorem</i> 13.
1, Or	2	$y + z : y + z + x :: BA : ba$. See Page 192.
Again	3	$y : \square BA :: y + x : \square ba$, per <i>Theorem</i> Page 380.
3, Or	4	$y : y + x :: \square BA : \square ba$
2 in $\square$'s	5	$\{ yy + 2yz + zz : yy + 2yz + 2yx + zz + 2zx + xx :: \square BA : \square ba$
4, 5	6	$\{ y : y + x :: yy + 2yz + zz : yy + 2yz + 2yx + 2zx + xx$
6	7	$\{ yy + 2yz + yx + zz + 2zx + \frac{zzx}{y} = yy + 2yz + 2yx + zz + 2zx + xx$
That is,	8	$\frac{zzx}{y} = yx + xx$, consequently $\frac{zz}{y} = y + x$
Suppose	9	$x = 0$ and rejected, as in the <i>Ellipsis</i> , Page 377.
Then	10	$\frac{zz}{y} = y$, consequently $zz = yy$
10 2	11	$z = y$, that is, $SP = SA$

Q. E. D.

C H A P. IV.

Concerning the chief Properties of the Hyperbola.

NOTE, any Part of the *Axis* of an *Hyperbola*, which is intercepted between its Vertex and any Ordinate (*viz.* any intercepted Diameter) is call'd an *Abscissa*; as in the *Parabola*.

SECT. I. *The Plain of every Hyperbola is proportion'd by this general Theorem.*

THEOREM. { As the Sum of the Transverse and any *Abscissa* multiply'd into that *Abscissa* : is to the Square of its Semi ordinate :: so is the Sum of the Transverse and any other *Abscissa* multiply'd into that *Abscissa* : to the Square of its Semi-ordinate.

That

That is, if TS be the Transverse Diameter,

And $\begin{cases} Sa, SA \text{ Abscissæ.} \\ ba, BA \text{ Semi-ordinates.} \end{cases}$

Then is $\begin{cases} Ta = TS + Sa \\ TA = TS + SA \end{cases}$

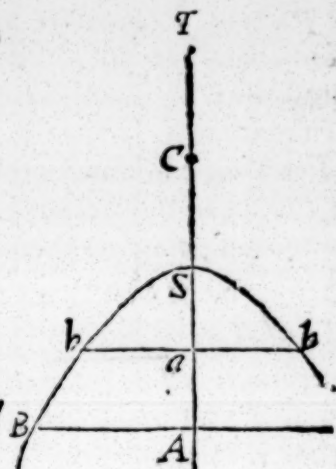
And it will be

$$Ta \times Sa : \square ba :: TA \times SA : \square BA$$

That is,

$$\overline{TS + Sa} \times Sa : \square ba :: \overline{TS + SA} \times SA : \square BA$$

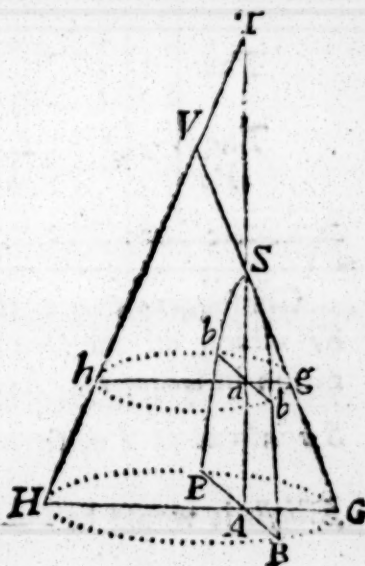
&c.



Demonstration.

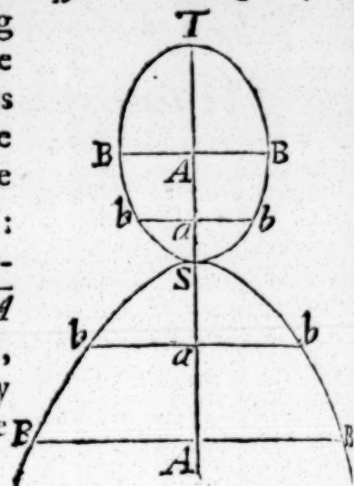
Let the following Figure HVG represent a *Right Cone* cut into two Parts by the Right Line SA ; then will the Plain of that Section be an *Hyperbola* (by *Seçt. 5, Chap 1.*) in which let SA be its Axis, or intercepted Diameter, bab and BA *Ordinates* rightly apply'd (as before in the *Parabola*) and TS its Transverse Diameter. Again, if the Cone is suppos'd to be cut by hg , parallel to its Base HG , it will also be the Diameter of a Circle, &c. as in the *Ellipsis* and *Parabola*. Then will the $\triangle Sga$ and $\triangle SGA$ be alike; also the $\triangle Tab$ and $\triangle TAH$ will be alike; therefore it

will be	1	$Sa : ag :: SA : AG$
And	2	$Ta : ab :: TA : AH$
1	3	$Sa \times AG = SA \times ag$
2	4	$Ta \times AH = TA \times ab$
3	5	$\begin{cases} Sa \times Ta \times AG \times AH = \\ SA \times TA \times ag \times ab \end{cases}$
But	6	$ag \times ab = \square ab$
And	7	$\begin{cases} AG \times AH = \square AB \\ \text{per Lemma Page 363.} \end{cases}$
5, 6, 7	8	$\begin{cases} Sa \times Ta \times \square AB = \\ SA \times TA \times \square ab \end{cases}$
8	9	$Sa \times Ta : \square ab :: SA \times TA : \square AB$ &c.



Q. E. D.

These *Proportions* are the common Property of every *Hyperbola*, and do only differ from those of the *Ellipsis* in the Signs + and —; as plainly appears in the following *Proportions*. That is, if we suppose *TS* the Transverse Diameter common to both Sections (*viz.* both the *Ellipsis* and *Hyperbola*) as in the annexed Scheme: then in the *Ellipsis* it will be $\overline{TS} - Sa \times Sa : \square ab :: \overline{TS} - SA \times SA : \square AB$ as by *Seet. 1, Chap. 2.* and in the *Hyperbola* it is $\overline{TS} + Sa \times Sa : \square ab :: \overline{TS} + SA \times SA : \square AB$, as above. Therefore all, that is farther requir'd in the *Hyperbola*, may (in a manner) be found as in the *Ellipsis*, due Regard being had to changing of the *Sine*.



Sect. 2. To find the Latus Rectum, or Right Parameter, of any Hyperbola.

From the last Proportion take either of the Antecedents and its Consequent, *viz.* either $Ta \times Sa : \square ab$. Or $TA \times SA : \square AB$, to them bring in the Transverse *TS* for a third Term, and by those three find a fourth Proportional (as in the *Ellipsis*) and that will be the *Latus Rectum*.

Thus	1	$\left\{ \begin{array}{l} Ta \times Sa : \square ab :: TS : \frac{\square ab \times TS}{Ta \times Sa} = \text{the Latus} \\ \text{Rectum, which call } L \text{ (as in the Parabola.)} \end{array} \right.$
Then	2	
But	3	$TS : L :: Ta \times Sa : \square ab$.
2.	3	$Ta \times Sa : \square ab :: TA \times SA : \square AB$, therefore
	4	$TS : L :: TA \times SA : \square AB$, &c.

Consequently *L* is the true *Latus Rectum*, or right Parameter, by which all the *Ordinates* may be found, according to its Definition in *Chap. 1.* And because $\overline{TS} + Sa = Ta$, let it be $\overline{TS} + Sa$ instead of *Ta*, then it will be $\frac{\square ab \times TS}{\overline{TS} \times Sa + \square Sa} = L$ and in the *Ellipsis* it would be $\frac{\square ab \times TS}{\overline{TS} \times Sa - \square Sa} = LR = L$.

Sect.

Sect. 3. To find the **Focus** of any Hyperbola.

The *Focus* being that Point in the *Hyperbola's* Axis through which the *Latus Rectum* must pass (as in the *Ellipsis* and *Parabola*) it may be found by this *Theorem*.

THEOREM. $\left\{ \begin{array}{l} \text{To the Rectangle made of half the Transverse into} \\ \text{half the Latus Rectum, add the Square of half the} \\ \text{Transverse; the Square Root of that Sum will be the} \\ \text{Distance of the Focus from the Center of the Hy-} \\ \text{perbola.} \end{array} \right.$

Demonstration.

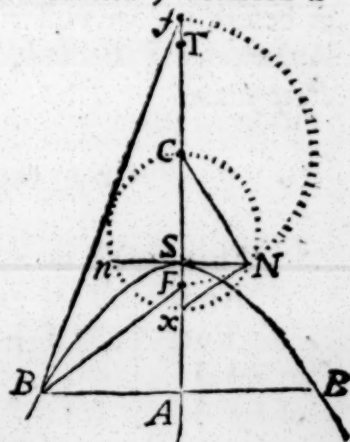
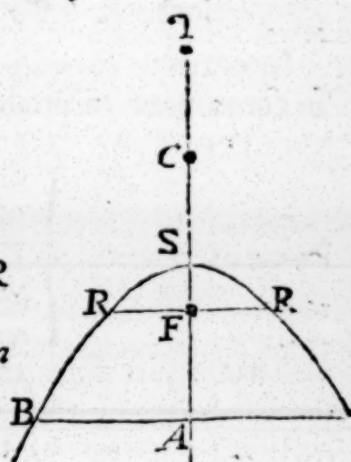
Suppose the Point at F , in the annex'd Scheme, to be the *Focus* sought; then will $FR = \frac{1}{2}L$. Let $TC = CS$ be half the Transverse; then is the Point C call'd the *Center* of the *Hyperbola* (for a Reason that shall be hereafter shew'd.) Again; let $CS = d$. and $SF = a$

Then	1	$2d : L :: \frac{2d+a}{2} \times a : \frac{1}{4}LL$
That is,	2	$TS : L :: TS + SF \times FS : \square FR$
1	3	$\frac{1}{2}dL = 2da + aa$
3 + dd	4	$dd + \frac{1}{2}dL = dd + 2da + aa$
4 ω^2	5	$\sqrt{dd + \frac{1}{2}dL} = d + a = FC$
Or 5, - d	6	$\sqrt{dd + \frac{1}{2}dL} - d = a = SF$

In the *Ellipsis* 'tis, $2d : L :: 2d - a \times a : \frac{1}{4}LL$. that is, $\frac{1}{2}dL = 2da - aa$, &c.

The Geometrical Effecti^{on} of the last Theorem is very easily perform'd, thus: make $Sx = \frac{1}{2}L$, viz. half the *Latus Rectum*; and let $CS = d$, as above. Upon Cx (as a *Diameter*) describe a Circle, and at S the *Vertex* of the *Hyperbola* draw the Right Line nSN at Right Angles to Cx ; then join the Points CN with a Right Line, and 'twill be $CN = d + a = FC$.

For	1	$CS : SN :: SN : Sx$, per Fig.
That is,	2	$d : SN :: SN : \frac{1}{2}L$.
2	3	$\frac{1}{2}dL = \square SN$
But	4	$dd + \square SN = \square CN$
3, 4	5	$dd + \frac{1}{2}dL = \square CN$
5 ω^2	6	$\sqrt{dd + \frac{1}{2}dL} = CN = d + a$, &c.



Now

Now here is not only found the Distance of the *Hyperbola's Focus*, either from its Center *C*, or Vertex *S*, but here is also found that Right Line usually call'd its Conjugate Diameter, *viz.* the Line $n\ S\ N$, which bears the same Proportion to the Transverse and *Latus Rectum* of the *Hyperbola*, as the Conjugate Diameter of the *Ellipsis* doth to its Transverse and *Latus Rectum*. For in the *Ellipsis* $TS : Nn :: Nn : LR$. per *Seet. 2, Page 363*. Consequently $\frac{1}{2} TS : \frac{1}{2} Nn :: \frac{1}{2} Nn : \frac{1}{2} LR$. But $\frac{1}{2} TS = d$, $\frac{1}{2} Nn = SN$, and $\frac{1}{2} LR = \frac{1}{2} L$. Therefore $d : SN :: SN : \frac{1}{2} L$. As at the 2d Step above.

What Use the aforesaid Line $n\ S\ N$ is of, in Relation to the *Hyperbola*, will appear farther on.

Sect. 4. To describe an Hyperbola in Plano.

In order to the easy Describing of an *Hyperbola in Plano*, it will be convenient to premise the following *Proposition*, which differs from that of the *Ellipsis* in *Seet. 3, Chap. 2*, only in the Signs.

PROPOSITION. $\left\{ \begin{array}{l} \text{If from the Foci of an Hyperbola there be drawn} \\ \text{two Right Lines, so as to meet each other in any} \\ \text{Point of the Hyperbola's Curve, the Difference of} \\ \text{those Lines (in the Ellipsis 'tis their Sum) will be} \\ \text{equal to the Transverse Diameter,} \end{array} \right.$

That is, if *F* be the *Focus*, and it be made $Cf = CF$ (as in the last Scheme) then the Point *f* is said to be a *Focus* out of the Section (or rather of the opposite Section) and it will be $fB - FB = TS$.

Demonstration.

Suppose fC , or $Cf = z$, and $SA = x$, let CS , or $TC = d$, as before; then will $fA = d + x + z$, and $FA = d + x - z$. Again, let $FB = b$, and $fB = b$, then $2d = b - b$, by the *Proposition*.

From these substituted Letters it follows,

That	1	$dd + 2dx + 2dz + xx + 2zx + zz = \square fA$
And	2	$dd + 2dx - 2dz + xx - 2zx + zz = \square FA$
But		$\square fA + \square AB = \square fB$, and $\square fA + \square AB = \square FB$
Per 4th of last	3	$dd + \frac{1}{2} dL = da + 2da + aa = \square FC = zz$.

3 — dd	4	$zz - dd = \frac{1}{2} d L$	
4 ÷ $\frac{1}{2} d$	5	$\frac{zz - dd}{\frac{1}{2} d} = L$	
Again	6	$2d : L :: 2d + x \times x : \square AB$, by common Properties.	
5, 6	7	$2d : \frac{zz - dd}{\frac{1}{2} d} :: 2dx + xx : \square AB$	
7	8	$\frac{2dzzx + zxxx - 2dddx - ddxx}{dd} = \square AB$	
8 + 8	9	$\frac{dd + 2dx + 2dz + xx + 2zx + zz + 2dzzx + zxxx - 2d^3x - ddxx}{dd} = \square fA + \square AB = bb$	
2 + 8	10	$\frac{dd + 2dx - 2dz + xx - 2zx + z + 2dzzx + zxxx - 2d^3x - ddxx}{dd} = \square FA + \square AB = bb$	
9 + d	11	$d^4 + 2d^3z + 2ddzx + ddzz + 2dzzx + zxxx = ddhb$	
10 × dd	12	$d^4 - 2d^3z - 2ddzx + ddzz + 2dzzx + zxxx = ddbb$	
11 w^2	13	$dd + dz + zx = db$	Although the Equation at the 16th Step be in itself impossible, because z is greater than d (by the 4th Step), yet from thence it will be easy to conclude, that the Difference between d and $z + \frac{zx}{d}$ will give the true Value of b, as in the 17th Step.
12 w^2	14	$dd - dz - zx = db$	
13 ÷ d	15	$d + z + \frac{zx}{d} = b$	
14 ÷ d	16	$d - z - \frac{zx}{d} = b$	
16, or	17	$z + \frac{zx}{d} - d = b$	
15 — 17	18	$2d = b - b$	

But because I would leave no Room for the Learner to doubt about changing the Equation, $d - z - \frac{zx}{d} = b$ into that of

$z + \frac{zx}{d} - d = b$, it may be convenient to illustrate the whole Process in Numbers, whereby (I presume) 'twill plainly appear that $b - b = TS$.

In order to that, let the Transverse $TS = 2d = 12$, then $d = 6$ suppose the Abscissa $SA = x = 4$, and the Semi-ordinate $AB = 3$.

First	1	$TS + SA \times SA : \square AB :: TS : L$, per Sect. 2.
1, viz.	2	$12 + 4 \times 4 = 64 : 9 :: 12 : 1,6875 = L$
Again	3	$\sqrt{dd + \frac{1}{2} dL} = d + a = CF$, per Sect. 3.
3, viz.	4	$\sqrt{36 + 5,0625} = 6,408 = CF = z$
Then	5	$d + x + z = 6 + 4 + 6,408 = 16,408 = fA$
And	6	$d + x - z = 6 + 4 - 6,408 = 3,592 = FA$

5	⊙ ²	7	269,2224 = $\square fA$
6	⊙ ²	8	12,9024 = $\square FA$
	But	9	9 = $\square AB$, for $AB = 3$ by Supposition.
7	+	10	278,2224 = $\square fA + \square AB = \square fB$
8	+	11	21,9024 = $\square FA + \square AB = \square FB$
10	w^2	12	16,68 = fB
11	w^2	13	4,68 = FB
12-13		14	12,00 = $fB = FB - TS$. Which was to be prov'd.

If this *Proposition* be truly understood, it must needs be easy to conceive how to describe the Curve of any *Hyperbola* very readily by *Points* when the *Transverse Diameter* and the *Focus* are given (or any other *Data* by which they may be found, as in the precedent Rules) thus :

Draw any straight Line at Pleasure, and on it set off the Length of the given *Transverse TS*, and from its extreme Points or Limits, viz. *TS*, set off $Tf = SF$, the Distance of the given *Focus* (viz. the Point *f* without, and *F* within the Section, as before) : that done, upon the Point *f* (as a Center) with any assum'd Radius greater than *TS*, describe an Arch of a Circle; then from that Radius take the Transverse *TS*, making their Difference a second Radius, with which, upon the Point *F* within the Section, describe another Arch to cut or cross the first Arch, as at *B*; then will that Point *B* be in the Curve of the *Hyperbola*, by the last Proposition. And therefore 'tis plain, that, proceeding on in this Manner, you may find as many Points (like *B*) as may be thought convenient (the more there are, and nearer they are together, the better) which being all join'd together with an even Hand (as in the Parabola) will form the *Hyperbola* requir'd.



There are several other Ways of delineating an *Hyperbola* in *Plano* : One Way is, by finding a competent Number of *Ordinates*, as by Section 1, &c. but I think none so easy and expeditious as this mechanical Way : I shall therefore, for Brevity's Sake, pass over the rest, and leave them to the Learner's Practice, as being easily deduced from what hath been already said.

Sect.

Sect. 5. To draw a Tangent to any given Point in the Curve of an Hyperbola.

The drawing of a Tangent, that will touch any given Point in the Curve of an Hyperbola, may be easily perform'd by Help of a Theorem; as in the Ellipsis, Sect. 6, Chap. 2.

Let $\begin{cases} D = TS \text{ the Transverse Diameter.} \\ L = \text{the Latus Rectum.} \\ y = SA \text{ the Abscissa.} \end{cases}$

And $z = AP$ $\begin{cases} \text{the Distance between the} \\ \text{Ordinate and that Point} \\ \text{in the Transverse cut by} \\ \text{the Tangent.} \end{cases}$

Then, if y be given, z may be found by this Theorem, $\begin{cases} \frac{Dy + yy}{\frac{1}{2}D + y} = z \end{cases}$ [which differs from that in the Ellipsis only in Signs. Vide Page 371.]

Or, if z be given, then y may be found by this Theorem:

$$\text{THEOREM. } \sqrt{\frac{DD + zz}{4}} : + \frac{1}{2}z - \frac{1}{2}D = y.$$

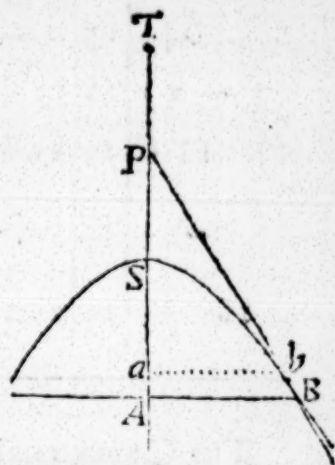
Demonstration.

Draw the Semi-ordinate ba , as in the Figure, and put $x = Aa$ $\begin{cases} \text{an infinite small Space between the two Semi-ordinates;} \\ \text{as before in the Ellipsis, \&c.} \end{cases}$

Then	1	$D : L :: Dy + yy : \square AB$
That is,	2	$TS : L :: TS + SA \times SA : \square AB$
1 ..	3	$\frac{DyL + yyL}{D} = \square AB$
Again	4	$D : L :: Dy + yy - 2yx - Dx + xx : \square ab$
That is,	5	$TS : L :: TS + Sa \times sa : \square ab$
4 ..	6	$\frac{DyL + yyL - 2yxL - DxL + xxL}{D} = \square ab$
Per Figure	7	$z : AB :: z - x : ab$, viz. $PA : AB :: Pa : ab$
7 in $\square$'s	8	$zz : \square AB :: zz - 2zx + xx : \square ab$
Suppose	9	$x = 0$ and every-where rejected (as in the Ellipsis)
Then 3, 9	10	$zz : \frac{DyL + yyL}{D} :: zz - 2zx : \square ab$
10 ..	11	$\frac{DyLzz + yyLzz - 2DyLz - 2yyLz}{Dzz} = \square ab$

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6, 11



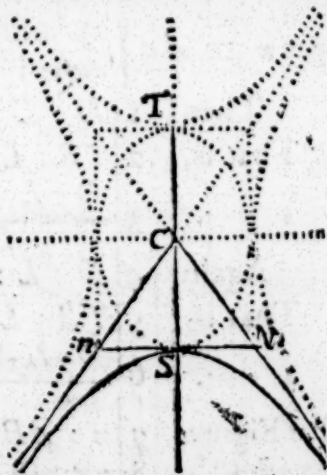
6 ₁	11	12	$\left\{ \begin{array}{l} \frac{DyL + yyL - 2yL - DL}{D} = \\ \frac{DyLzz + yyLzz - 2DyLz - 2yyLz}{Dzz} \end{array} \right.$
12 reduced	13		$Dz + zy = Dy + yy$
13 Analogy	14		$D + y : y :: D + y : z$, viz. $CA : SA :: TA : AP$
13 $\div \frac{1}{2}D + y$	15		$z = \frac{Dy + yy}{\frac{1}{2}D + y}$ which is the first Theorem.
13 $- zy$	16		$yy + Dy - zy = \frac{1}{2}Dz$
16 $\square C$	17		$yy + Dy - zy + \frac{DD - 2Dz + zz}{4} = \frac{DD + zz}{4}$
17 w^2	18		$y + \frac{1}{2}D - \frac{1}{2}z = \sqrt{\frac{DD + zz}{4}}$
18 $\pm$	19		$y = \sqrt{\frac{D + zz}{4}} : + \frac{1}{2}z - \frac{1}{2}D \left\{ \begin{array}{l} \text{which is the se-} \\ \text{cond Theorem.} \end{array} \right.$

Q. E. D.

The Geometrical Effecton of the first of these Theorems is very easy ; for, by the 14th Step, 'tis evident that there are three Lines given to find a fourth proportional Line. By Problem 3, Page 308.]

Scholium.

From the Comparifons, which have been all-along made in this Chapter, between the *Hyperbola* and the *Ellipsis*, 'twill be easy (even for a Learner) to perceive the Coherence that is in (or between) those two Figures ; but, for the better understanding of what is meant by the *Center* and *Asymptotes* of an *Hyperbola*, consider the annex'd Scheme, wherein it is evident (even by Inspection) that the opposite *Hyperbola*'s will always be alike, because they will always have the same Transverse Diameter common to both, &c. (see Sect. 1, of this Chap.) Also, that the middle Point, or common Center of the *Ellipsis*, is the common Center to all the four *conjugat Hyperbola*'s.



And the two Diagonals of the Right-angled *Parallelogram*, which circumscribes the *Ellipsis* (or is inscrib'd to the four *Hyperbola*'s) being continued, will be such *Asymptotes* to those *Hyperbola*'s as are defined Chap. 1, Sect. 5, Defin. 4.

Sect.

Sect. 6. To draw the Asymptotes of any Hyperbola, &c.

Having found the *Latus Rectum* (by Sect. 2.) and the Conjugate Diameter in nSN in its true Position, by Sect. 3. Then through the Center C of the Hyperbola, and the extream Points nN of its Conjugate Diameter, draw two Right Lines, as CN and Cn , infinitely continued (as in the following Figure) and they will be the *Asymptotes* required. That is, they are two such Right Lines as, being infinitely extended, will continually incline to the Sides of the Hyperbola, but never touch them.

Demonstration.

Suppose the *Semi-ordinates* ab and AB to be rightly apply'd to the Axis TA ; and produced both Ways to the *Asymptotes*, as at fg and FG ; then will the $\triangle CSN$, $\triangle Cag$, and $\triangle CAG$ be alike.

Let $d = CS = TC$. And $L =$ the *Latus Rectum*; as before,

Put $\begin{cases} e = Sa \\ y = SA \end{cases}$ the *Abscissæ*. Then $\begin{cases} d + e = Ca \\ d + y = CA \end{cases}$

Then 1 $d : SN :: d + e : ag$. viz. $CS : SN :: Ca : ag$

1 in $\square$'s 2 $dd : \square SN :: dd + 2de + ee : \square ag$

But 3 $d = \square SN$. per Sect. 3.

2, 3 $\therefore$ 4 $\frac{ddL + 2deL + eeL}{2d} = \square ag$

Again 5 $2d : L :: 2de + ee : \square ab$, per Sect. 2.

5 $\therefore$ 6 $\frac{2deL + eeL}{2d} = \square ab$

4 — 6 7 $\frac{dL}{2} = \square ag - \square ab$

But $\begin{cases} 8 ag + ab = bf \\ 9 ag - ab = bg \end{cases}$ per Fig.

8 $\times$ 9 10 $\square ag - \square ab = bf \times bg$

7, 10 11 $bf \times bg = dL$

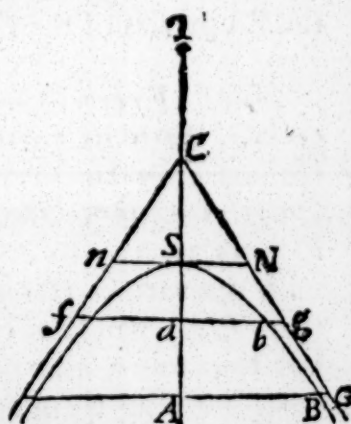
Again 12 $dd : \square SN :: dd + 2dy + yy : \square AG$

That is, $\square CS : \square SN :: \square CA : \square AG$

3, 12 $\therefore$ 13 $\frac{ddL + 2dyL + yyL}{2d} = \square AG$

But 14 $2d : L :: 2dy + yy : \square AB$, per Sect. 2.

14 $\therefore$ 15 $\frac{2dyL + yyL}{2d} = \square AB$



$$\begin{array}{rcl}
 13 - 15 & 16 & \frac{dL}{2} = \square AG - \square AB \\
 \text{Also } \left\{ \begin{array}{l} 17 \\ 18 \end{array} \right. & & \left\{ \begin{array}{l} AG + AB = BF \\ AG - AB = BG \end{array} \right\} \text{ per Fig.} \\
 17 \times 18 & 19 & \square AG - \square AB = BF \times BG \\
 16 & 19 & 20 \quad BF \times BG = \frac{1}{2} dL \\
 11, \& 20 \therefore & 21 \quad bg = \frac{\frac{1}{2} dL}{bf}. \quad \text{And } BG = \frac{\frac{1}{2} dL}{BF}
 \end{array}$$

From the last Step 'tis evident, that the *Asymptotes* are nearer the *Hyperbola* at *G* than at *g*, and consequently will continually approach to its Curve: For $BF) \frac{1}{2} dL (= BG$ is less than $bf) \frac{1}{2} dL (= bg$, because the *Divisor* BF is greater than the *Divisor* bf ; and it must needs be so where-ever the *Ordinates* are produc'd to the *Asymptotes*, from the Nature of the *Triangles*.

Again; From the 7th and 16th Steps 'tis evident, that the *Asymptotes* can never really meet and be co-incident with the Curve of the *Hyperbola*, although both were infinitely extended, because $\frac{1}{2} dL$ will always be the Difference between the Square of any *Semi-ordinate* and the Square of that *Semi-ordinate*, when 'tis produc'd to the *Asymptote*.

Confectary.

From hence it follows, that every *Right Line* which passes thro' the *Center* and falls within the *Asymptotes*, will cut the *Hyperbola*; and all such *Lines* are call'd *Diameters* (as in the *Ellipsis*) because the Properties of the *Hyperbola* and *Ellipsis* are the same.

Note. Every *Diameter*, both in the *Ellipsis*, *Parabola*, and *Hyperbola*, hath its particular *Latus Rectum* and *Ordinates*; which (should they be distinctly handled, and the Effect of all such *Lines* as relate to them, as also the Nature and Properties of such Figures as may be inscribed and circumscribed to all the Sections, with the various Habitues or Proportions of one *Hyperbola* to another, &c.) would afford Matter sufficient to fill a large Volume. But thus much may suffice by way of *Introduction*; I shall therefore desist pursuing them any farther, being fully satisfied, that, if what I have already done be well understood, the rest must needs be very easy to any one that intends to proceed farther on that subject.

A N

INTRODUCTION

T O T H E

Mathematicks.

P A R T V.

THE Method of finding out any particular Quantity (*viz.* either any LINE, SUPERFICIES, or SOLID) by a regular Progression, or Series of Quantities continually approaching to it, which, being infinitely continued, would then become perfectly equal to it; is what is commonly call'd *Arithmetick of Infinites*; which I shall briefly deliver in the following *Lemma's*, and apply them to Practice in finding the superficial and solid Contents of Geometrical Figures farther on.

L E M M A I.

If any Series of equal Numbers (representing Lines or other Quantities) as, 1. 1. 1. 1. &c. or 2. 2. 2. 2. &c. or 3. 3. 3. &c. if one of the Terms be multiply'd into the Number of Terms, the Product will be the Sum of all the Terms in the Series.

This is so very plain, and easy to be understood, that it needs no *Example*.

L E M M A II.

If the Series of Numbers in Arithmetick Progression begin with a Cypher, and the common Difference be 1; as, 0. 1. 2. 3. 4. &c. (representing a Series of Lines or Roots beginning with a Point) if the last Term be multiply'd into the Number of Terms, the Product will be double the Sum of all the Series.

That is, putting L = the last Term, N = the Number of Terms, and S = the Sum of all the Series:

Then

Then will $NL = 2S$. Consequently, $\frac{1}{2}NL = S$.
viz. one Half of so many times the greatest Term as there are Numbers of Terms in the Series.

Thus $\frac{0+1+2+3+4}{4+4+4+4+4} = \frac{10}{20} = \frac{1}{2}NL$.
 $\frac{10}{20} = S$.

And this will always be so, how many Terms soever there are, by *Consect. 1, Page 185*.

LEMMA III.

If a Series of Squares whose Sides or Roots are in Arithmetick Progression, beginning with a Cypher, &c. (as in the last Lemma) be infinitely continued; the last Term, being multiply'd into the Numbers of Terms, will be Triple to the Sum of all the Series, *viz.* $NLL = 3S$, or $\frac{1}{3}NLL = S$.

That is, the Sum of such a Series will be one Third of the last or greatest Term, so many times repeated as is the Number of Terms in the Series.

Instances in the Square Numbers.

1. $\left\{ \frac{0+1+4}{4+4+4} = \frac{5}{12} = \frac{1}{3} + \frac{1}{12} \right.$
2. $\left\{ \frac{0+1+4+9}{9+9+9+9} = \frac{14}{36} = \frac{7}{18} = \frac{1}{3} + \frac{1}{18} \right.$
3. $\left\{ \frac{0+1+4+9+16}{16+16+16+16} = \frac{30}{64} = \frac{3}{8} = \frac{9}{24} = \frac{1}{3} + \frac{1}{24} \right.$ &c.

From these Instances 'tis evident, that as the Number of Terms in the Series does increase, the Fraction or Excess above does decrease, the said Excess always being $\frac{1}{6N-6}$; which, if we suppose the Series to be infinitely continued, will then become infinitely small, *viz.* in Effect nothing at all. Consequently, NLL may be taken for the true or perfect Sum of such an infinite Series of Squares.

LEMMA IV.

If a Series of Cubes, whose Roots are in Arithmetick Progression, beginning with a Cypher, &c. (as above) be infinitely continu'd, the Sum of all the Series will be $\frac{1}{4}NLLL = S$.

That is, one Fourth of the last or greatest Term so many times repeated as is the Number of Terms.

Instances

Instances in Cube Numbers.

If 0 . 1 . 2 . 3. &c. be the Roots of the Cubes.

Then 1. $\left\{ \frac{0+1+8+27}{27+27+27+27} = \frac{36}{108} = \frac{4}{12} = \frac{1}{4} + \frac{1}{12} \right.$

2. $\left\{ \frac{0+1+8+27+64}{64+64+64+64+64} = \frac{100}{320} = \frac{10}{32} = \frac{5}{16} = \frac{1}{4} + \frac{1}{16} \right.$

3. $\left\{ \frac{0+1+8+27+64+125}{125+125+125+125+125+125} = \frac{225}{750} = \frac{45}{150} = \frac{3}{10} = \frac{6}{20} = \frac{1}{4} + \frac{1}{20} \right.$

From these *Examples* it plainly appears, that, as the Number of Terms in the Series increases, the Fraction or Excess above $\frac{1}{4}$ decreases, the Excess being always $\frac{1}{4N-4}$; which, if we suppose the Series to be infinitely continued, will become infinitely small, or rather nothing; as in the last *Lemma*. Consequently, $\frac{1}{4} NL$ may be taken for the true and perfect Sum of all the Terms in such an infinite Series of Cubes.

LEMMA V.

If a Series of Biquadrates, whose Roots are in Arithmetick Progression, beginning with a Cypher, &c. (as before) be infinitely continued, the Sum of all the Terms in such a Series will be $\frac{1}{5} NL^4$.

The Truth of this may be manifested by the like Process, as in the foregoing *Lemma's*, and so on for higher Powers. But if any one desires a farther Demonstration of these Series, he may (I presume) meet with ample Satisfaction in Dr. Wallis's History of Algebra, Chap. 78 and 79, wherein the Doctor concludes with these Words:

“ Thus having shew'd, that in the Progression of Laterals (or Arithmetical Proportionals) beginning at 0, the Sum of 2, 3, 4, 5, 6 Terms, is always equal to half of so many times the greatest; and there being no Pretence of Reason why we should then doubt it in a Progression of 7, 8, 9, 10, &c. we conclude it so to be, tho' such Number of Terms be suppos'd infinite.

“ Again; in a Progression of their Squares having shew'd, that in 2, 3, 4, 5, 6 Terms the Aggregate is always more than one Third of so many times the greatest, and the Excess always such

“ aliquot

" aliquot Part of the greatest, as is denominated by six times the
 " Number of Terms wanting 1. (As, if the Terms be 2,
 " it is $\frac{1}{3} + \frac{1}{6}$; if three, it is $\frac{1}{3} + \frac{1}{12}$; if 4, it is $\frac{1}{3} + \frac{1}{16}$; if 5,
 " it is $\frac{1}{3} + \frac{1}{24}$ of so many times the greatest Term, and so onward)
 " we may well conclude (there being no Pretence of Reason
 " why to doubt it in the rest) that it will be so, how many soever
 " be such Number of Terms. And because such Excess, as the
 " Number of Terms do increase, will become infinitely small (or
 " less than any assignable) we conclude (from the Method of Ex-
 " haustions) that, if the Number of Terms be suppos'd infinite,
 " such Excess must be suppos'd to vanish, and the Aggregate of
 " such infinite Progression suppos'd equal to $\frac{1}{3}$ of so many times
 " the greatest.

" In like manner having prov'd that such Progression of Cubes
 " doth (as the Number of Terms increase) approach infinitely near
 " to $\frac{1}{4}$ of so many times the greatest, and of Biquadrates to $\frac{1}{5}$, and
 " so of Surfolids to $\frac{1}{6}$ of so many times the greatest, and so on-
 " wards as we please to try; and there being no Pretence of Rea-
 " son why to doubt it as to the rest, we may take it as a sufficient
 " Discovery, that (universally) the Aggregate of such infinite
 " Progression is equal (or doth approach infinitely near) to such
 " a Part of so many times the greatest, as is denominated by the
 " Exponent (or Number of Dimensions) of such Power (as is
 " that according to which the Progression is made) increas'd by
 " 1, namely, of Laterals $\frac{1}{2}$; of Squares $\frac{1}{4}$; of Cubes $\frac{1}{6}$; of Bi-
 " quadrates $\frac{1}{5}$; of so many times the greatest) and so onwards
 " infinitely."

This Discourse of the Doctor's I thought convenient to insert,
 supposing it may give some Satisfaction to the Learner, to hear so
 Great a Man as Dr. Wallis's Argument about the Truth of these
 Series, which I have briefly deliver'd in the foregoing *Lemma's*.

LEMMA VI.

If any two Series or Ranks of Proportionals have the same Num-
 ber of Terms (whether Finite or Infinite) it will always

be { As the first Term of one Series : is to the first Term of the
 other Series :: so is the Sum of all the Terms in the one Series :
 to the Sum of all the Terms in the other Series.

(12. 4. 5)

As

As in these Numbers,	3	6	Or these Numbers,	4	5
	2	6		12	15
	3	9		36	45
	4	12		108	135
	5	15		324	405
	6	18		972	1215

That is, $1 : 3 :: 21 : 63$ And $4 : 5 :: 1456 : 1820$ &c.
 The Application of these *Lemma's* to Geometrical Quantities,
viz. to Lines, Superficies, and Solids, wholly depends upon grant-
 ing the following *Hypotheses*.

The Hypotheses.

1. That every Line is suppos'd to consist (or be compos'd) of an infinite Series of equidistant Points.

2. A Surface (*viz.* the Area of any Figure) to consist of an infinite Series of Lines, either streight or crooked, according as the Figure requires.

3. A Solid to consist of an infinite Series of Plains, or Superficies, according as its Figure requires.

Not that we suppose Lines, which have really no Breadth, can fill a Space or Superficies; or that Plains, which have not any Thickness, can constitute a Solid: But by what we here call Lines are to be understood small Parallelograms (or other Superficies) infinitely narrow, yet so as that their Breadths, being all taken and put together, must be equal to the Figure they are suppos'd to fill up. And those Plains or Superficies, which are here said to constitute a Solid, are to be understood infinitely thin; yet so as that their Depths or Thicknesses (which are hereafter, also called Lines) being all taken together, must be equal to the Height of the propos'd Solid. Now, in order to render this Hypothesis as easy for a Learner to understand as I can, I shall here propose a very plain and familiar *Example*; *Viz.* Let us suppose any Book to be compos'd (or made up) of 100, 200, 300 (more or less) Leaves of fine Paper; such a Book, being close put together, will have Length, Breadth, and Depth or Thickness, and therefore may (not improperly) be called a Solid; and each of its Edges (being evenly cut) will be a Superficies compos'd of a Series of small Parallelograms, every one of their Breadths being only the Edge of a single Leaf of Paper; and if we conceive the Thickness of every one of those Leaves to be divided into 10, or

F f f

100,

“ aliquot Part of the greatest, as is denominated by fix times the
 “ Number of Terms wanting 1. (As, if the Terms be 2,
 “ it is $\frac{1}{3} + \frac{1}{6}$; if three, it is $\frac{1}{3} + \frac{1}{12}$; if 4, it is $\frac{1}{3} + \frac{1}{16}$; if 5,
 “ it is $\frac{1}{3} + \frac{1}{24}$ of so many times the greatest Term, and so onward)
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 “ why to doubt it in the rest) that it will be so, how many soever
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 “ Number of Terms do increase, will become infinitely small (or
 “ less than any assignable) we conclude (from the Method of Ex-
 “ haustions) that, if the Number of Terms be suppos'd infinite,
 “ such Excess must be suppos'd to vanish, and the Aggregate of
 “ such infinite Progression suppos'd equal to $\frac{1}{3}$ of so many times
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“ In like manner having prov'd that such Progression of Cubes
 “ doth (as the Number of Terms increase) approach infinitely near
 “ to $\frac{1}{4}$ of so many times the greatest, and of Biquadrates to $\frac{1}{5}$, and
 “ so of Surfolids to $\frac{1}{6}$ of so many times the greatest, and so on-
 “ wards as we please to try; and there being no Pretence of Rea-
 “ son why to doubt it as to the rest, we may take it as a sufficient
 “ Discovery, that (universally) the Aggregate of such infinite
 “ Progression is equal (or doth approach infinitely near) to such
 “ a Part of so many times the greatest, as is denominated by the
 “ Exponent (or Number of Dimensions) of such Power (as is
 “ that according to which the Progression is made) increas'd by
 “ 1, namely, of Laterals $\frac{1}{2}$; of Squares $\frac{1}{4}$; of Cubes $\frac{1}{6}$; of Bi-
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100, or 1000, &c. they will then become such a Series of infinitely small Lines as are (by the Hypothesis) said to compose or fill up a Superficies. And all the Superficies of those infinitely thin or divided Leaves of Paper will become such a Series of Plains, or Superficies, as are said to constitute a Solid, viz. such a Solid as the Bigness and Figure of that Book.

Now according to this Idea of Lines, Superficies, and Solids, one may, without the least Prejudice to any *Demonstration*, admit of the following *Definitions* and *Theorems*.

Definitions.

I. The Area's of Squares, and all other Parallelograms, are compos'd or fill'd up with an infinite Series of equal Right Lines.

II. The Area of every *plain Triangle* is compos'd of an infinite Series of Right Lines parallel to its Base, and equally decreasing until they terminate in a Point at the vertical Angle.

III. The Area of a *Circle* may be composed either of an infinite Series of concentrick or parallel Circles, or of an infinite Series of Chord Lines parallel to its Diameter, or of an innumerable Multitude of Sectors.

IV. The Area of an *Ellipsis* may be compos'd either of an infinite Series of Ordinates rightly apply'd, or of an infinite Series of Right Lines parallel to its Transverse Diameter.

V. The Area's of the *Parabola* and *Hyperbola* are compos'd of an infinite Series of Ordinates; or may also be compos'd of Right Lines parallel to its Axis, &c.

VI. A *Prism* is a solid Body contain'd or included within several equal Parallelograms, having its Bases or Ends equal and alike; and it is generally nam'd according to the Figure of its Base: That is,

VII. A *Cube* (or Solid like a Dye) is a Prism bounded or included with six equal square Plains.

VIII. A *Parallelopipedon* is a Prism that hath its Sides bounded or included within four equal Parallelograms and two square Bases or Ends.

IX. A *Cylinder* (or Solid, like a Rolling-stone in a Garden) is only a round Prism, having its Bases or Ends a perfect Circle.

X. The

X. The Solidity of every Prism is compos'd of an infinite Series of equal Plains, parallel and alike to that of its Base.

XI. A *Pyramid* is a Solid bounded or included within several plain Triangles set upon any Polygonous Base, having their vertical Angles all meeting together in a Point, called the Vertex, and takes its Name from the Figure of its Base, viz. if it has a square Base, 'tis call'd a square Pyramid; if a triangular Base, 'tis call'd a triangular Pyramid, &c.

XII. A *Cone* is only a round Pyramid, which hath been already defined in Page 355, &c.

XIII. The Solidity of every Pyramid is compos'd or constituted of an infinite Series of Plains, parallel and alike to that of its Base, equally decreasing until they terminate in a Point at the Vertex.

XIV. A *Sphere* or *Globe*, (viz. a Ball) is a Solid bounded or included within one regular Superficies, being form'd or generated by the Rotation of a Semi-circle about its Diameter (call'd the Axis of a Sphere) and its Solidity is compos'd or constituted of an infinite Series of concentrick Circles, whose Diameters are the Chords of that Circle by which it was form'd.

XV. A *Spheroid* (or Egg-like Figure) is a Solid bounded with one regular Superficies, form'd by the Rotation of a Semi-ellipsis about its Transverse Diameter (call'd the Axis of the Spheroid) and its Solidity is constituted of an infinite Series of concentrick Circles, whose Diameters are the Ordinates of that Ellipsis by which it was form'd.

XVI. There is another Sort of Solid call'd an *Oblate Spheroid*, being formed by the Rotation of an Ellipsis about its Conjugate Diameter, and it is like a flat Turnep.

XVII. If a Semi-parabola be turn'd about its Axis, 'twill form a Solid call'd a *Parabolick Conoid*, being compos'd or constituted of an infinite Series of Circles, whose Diameters are the Ordinates of a Parabola.

XVIII. If a Parabola be turn'd about its Base, or greatest Ordinate, 'twill form a Solid call'd a *Pyramidoid*, but most commonly a *Parabolick Spindle*, which will be constituted of an infinite Series of Circles, whose Diameters are Right Lines parallel to the Parabola's Axis.

XIX. If an Hyperbola be turn'd about its Axis, 'twill form a Solid call'd an *Hyperbolick Conoid*, being constituted of an infinite Series of Circles, whose Diameters are the Ordinates of the Hyperbola.

XX. The curve Superficies of all circular Solids (*viz.* Cylinders, Cones, Spheres, &c.) are compos'd of an infinite Series of the Peripheries of those Circles which constitute their Solidities.

Upon these Definitions are grounded all the following *Theorems*; and therefore, if they were diligently compar'd with their respective Figures, it must needs be of great Help to the Learner, and would render all that follows very easy; wherein I shall begin with what hath been already demonstrated, by way of introducing the rest.

THEOREM I.

The Area of every Right-angled Parallelogram is obtain'd by multiplying the Length into its Breadth.

That is, $BD \times FB =$ the Area of the Parallelogram $BDFG$, by *Lemma I*, compar'd with Definition I.

Example.

Suppose $BD = 26$, and $FB = 9$,
then $26 \times 9 = 234$ the Area.
See *Prob. I*, Page 339.

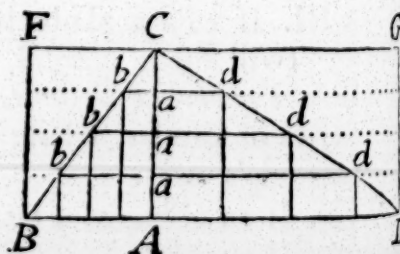


THEOREM II.

The Area of every plain Triangle is equal to half the Area of its circumscribing Parallelogram. That is, $\frac{BD \times CA}{2} =$ the Area of $\triangle BCD$, in the following Figure.

Demonstration.

Suppose the Perpendicular CA to be divided into an infinite Number of equal Parts, as at the Points a, a, a , &c. and through those Points there were drawn Right Lines parallel to the Base BD ; (*viz.* bad, bad, bad , &c.) then will those Lines be a Series of Terms in Arithmetick Progression beginning at the Point C (*viz.* $0, bd, 2bd, 3bd$, &c. as is evident by the Figure, wherein BD the greatest Term $= L$, and CA the Number of Terms $= N$.



But $\frac{1}{2} NL = S$, by Lemma 2. And $S =$ the Triangle's Area by Definition 2. Q. E. D.

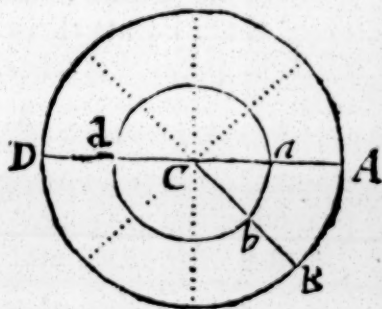
Example. Let $BD = 26$, and $CA = 9$, as above; then $\frac{26 \times 9}{2} = 117$, or $6\frac{1}{2} \times 9 = 117$. Or thus, $26 \times \frac{9}{2} = 117$, the Area requir'd. [See Problem 3, Page 330.]

THEOREM III.

The Peripheries of Circles are in Proportion one to another as their Diameters are.

Demonstration.

Let the Periphery of a Circle be divided into any Number of equal Arches by Right Lines drawn from the Center (*viz.* Radii) suppose 'em 8, as in the annexed Figure, wherein AB is one of them; then, if thro' any Point in the Radius there be drawn a concentrick or parallel Circle, its Periphery will also be divided into 8 equal Arches by those Radii, one whereof will be ab , and the $\triangle Cab$ will be like to $\triangle CAB$, Therefore $Ca : ab :: CA : AB$, or $Ca : CA :: ab : AB$, consequently $2Ca : 2CA :: 8ab : 8AB$. But $2Ca = da$ the Diameter of the Circle, whose Periphery is $8ab$; and $2CA = DA$, the Diameter of the Circle, whose Periphery is $8AB$. Therefore, &c. as by the Theorem. Q. E. D.



Example.

In Chapter 6, Part III, it was found, that, if the Diameter of a Circle be 2, its Periphery will be 6,2831853, &c. Therefore, $2 : 6,2831853, \&c. :: 1 : 3,14159265, \&c.$ the Periphery of the Circle whose Diameter is 1.

Corollary.

Hence it follows, that because Unity, or 1, may be made the first Term in the Proportion, therefore 3,14159265, &c. may be made a constant or settled Factor; which, being multiply'd into any propos'd Diameter, will produce the Periphery of that Circle.

Note, Instead of 3,14159265, &c. it may be sufficient to take only 3,1416.

Or,

Or, in whole Numbers, the Proportion may be,
 As 7 : 22 :: Diam. : Periphery } these Numbers may serve,
 Or 113 : 355 :: Diam. : Periphery } and are often used in com-
 mon Practice.

THEOREM IV.

The Area of any Sector of a Circle is equal to half the Rectangle of the Radius into its Arch. That is, $\frac{CA \times AB}{2} =$ the Area of ACP .

Demonstration.

Suppose the Radius CA to be divided into an infinite Series of equidistant Points, as a, e, y , &c. and thro' those Points there were drawn concentrick or parallel Arches, as ab, ed, yf , &c. then they will be a Series of Arches in Arithmetick Progression, beginning at the Point C (*viz.* 0, 1, 2, 3, &c.) as plainly appears by the Figure, wherein the greatest Term is $AB = L$, and Number of Terms is $CA = N$. But $\frac{1}{2} NL = S$, the Sum of all the Series, by Lemma 2, and $S =$ the Sector's Area, by Definition 3. Q. E. D.



Let the Radius $CA = 12$, and the Arch $AB = 8$, then $\frac{12 \times 8}{2} = 48$. Or $\frac{1}{2} \times 8 \times 12 = 48$. Or $\frac{8}{2} \times 12 = 48$, the Area of the Sector ACB .

THEOREM V.

The Area of every Circle is equal to half the Rectangle of the Radius into its Periphery. That is, according to Archimedes, a Circle is equal to a Right-angled Triangle, whose Sides containing the Right-angle are equal, one to the Radius, and the other to the Perimeter of that Circle. Pro. 1. de Dimensione Circuli.

The Truth of this Theorem may be easily deduced from the last, thus: If we suppose the last Sector to be one Eighth-part of a Circle, then it follows, that $\frac{8 AB \times CA}{2} = 4 AB \times CA$ will be the Area of the whole Circle. But $4 AB =$ half the Circle's Periphery, and $CA =$ half its Diameter; therefore, &c. as per Theorem.

Q. E. D.

Example.

Example.

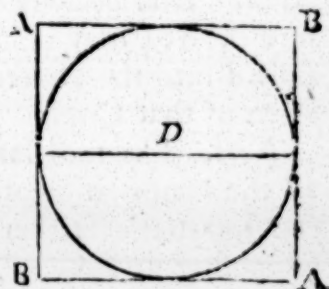
If the Diameter be Unity, or 1, the Periphery will be 3,14159265
&c. by Theorem 3. Then $\frac{3,14159265}{2} \times \frac{1}{2} = 0,78539816$,
&c. (or 0,7854 for common Use) will be the Area of that Circle.

Scholium.

From hence naturally flows the following Proportion between
the Square and its inscrib'd Circle.

PROPORTION. { As the Perimeter (*viz.* the Sum of the four Sides)
of any Square : is to its Area :: so is the Peri-
phery of the inscrib'd Circle : to its Area.

That is, supposing $AB = D =$ the Side of the Square, and
the Diameter of its inscrib'd Circle ;
then $4 D =$ the Perimeter, $DD =$ the
Area of the Square, and $3,1416 D =$ the
Periphery of the Circle, by Theorem 3.
But $4 D : DD :: 3,1416 D : 0,7854 DD$
 $=$ the Circle's Area. And if $D = 1$,
then $4 D = 4$, and $DD = 1 \times 1 = 1$,
and the Periphery will be 3,1416. Then
 $4 : 1 :: 1 : 0,7854$ &c. as in the Example
above. And from hence may be easily
deduced the following Theorems.



THEOREM VI.

The Area's of all Circles are in Proportion one to another as the
Squares of their Diameters. (2. e. 12.)

For if $D =$ the Diameter of one Circle, and $d =$ the Diameter
of another Circle, then will $0,7854 DD$ be the Area of one
Circle, and $0,7854 dd$ will be the Area of the other Circle ; as
above. But $0,7854 DD : 0,7854 dd :: DD : dd$. Or thus, let D
 $=$ the Diameter, and $P =$ the Periphery of one Circle ; $d =$ the
Diameter, and $p =$ the Periphery of another Circle ;

Then $1 \mid \frac{1}{2} D \times \frac{1}{2} P = \frac{1}{4} DP = A$, the Area of one Circle.

And $2 \mid \frac{1}{2} d \times \frac{1}{2} p = dp = a$, the Area of the other Circle.

$$1 \times 4 \mid 3 \mid DP = 4A$$

(per last Theorem.

$$2 \times 4 \mid 4 \mid dp = 4a$$

$$3 \div D \mid 5 \mid P = \frac{4A}{D}$$

$$4 \div d$$

			<i>Prob. 4. P being given to find A.</i>	
2 $\odot^2$	6	9,86965	$DD = PP$	
6 $\div$	7	$DD = \frac{PP}{9,86965}$	or 0,10132	$PP = DD$
4 $\div$	8	$DD = \frac{A}{0,7854}$	or 1,2732	$A = DD$
For		$\frac{1}{0,7854} = 1,2732$		
7,	8	9	$\frac{PP}{9,86965} = \frac{A}{0,7854}$	or 0,10132 $PP = 1,2732 A$
9 $\times$ &c.	10	$\frac{PP}{12,5664} = A$	or 0,07957	$PP = A$
<i>Prob. 5. A being given, to find D.</i>				
8 ω^2	11	$D = \sqrt{\frac{A}{0,7854}}$	or $D = \sqrt{1,2732 A}$	
<i>Prob. 6. A being given, to find P.</i>				
10 $\times$ &c.	12	$PP = 12,5664 A$	or $PP = \frac{A}{0,07957}$	
12 ω^2	13	$P = \sqrt{12,5664 A}$	or $R = \sqrt{\frac{A}{0,07957}}$	

These six Problems contain all the Variety that can be proposed about finding the Periphery, Diameter, and Area of any Circle.

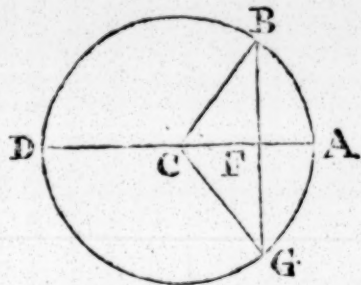
But if it be required to find the Area of any Segment, or Part of a Circle cut off by a Chord, that Work will require a farther Consideration.

First, As to the Data there must always be given the Diameter; or, either the Periphery or Area of the Circle, in order to find the Diameter.

Secondly, There must also be given, either the Chord which is the Base of the Segment, or the versed Sine, which is the Height of the Segment. That is, either BG, or AF, in the following Scheme, must be given, that so the Area of the $\triangle BCG$ may be found. Then it's evident (by the Figure) that, if the Area of the $\triangle BCG$ be taken from the Area of the Sector CBAG, the Remainder will be the Area of the Segment BAG. And if the Area of the Segment BAG be taken from the whole Area of the Circle, the Remainder will be the Area of the other Segment DBG.

Example in Numbers.

Let there be given $DA = 32$, as in *Prob. 1.* and the *versed Sine* $AF = 6$; then $\frac{1}{2} DA = BC = CA = 16$, and $CA - AF = CF = 10$. But $\square BC - \square CF = \square BF$. Consequently $\sqrt{\square BC - \square CF} = BF$, viz. $\sqrt{156} = 12,49 = BF$.



Then, by the Doctrine of plain Triangles, the Arch $BA = \angle BCA$ may be found in Degrees and Decimal Parts. Thus $BC : Radius :: BF : \text{Sine } \angle BCF = 51,31$ Degrees. And then it will always hold in this Proportion;

Viz. $\left\{ \begin{array}{l} \text{As the Circle's Periphery in Degrees: is to its Periphery in} \\ \text{equal Parts (according to the Dimenſions taken): : So is} \\ \text{the Arch in Degrees (viz. } \angle BCA \text{): to the same Arch in} \\ \text{equal Parts.} \end{array} \right.$

That is, $360^\circ : 100,5312 :: 51,31^\circ : 14,3284 = BA$. Then $14,3284 \times 16 = 229,2544$, the Area of the Sector $BCAG$; and $12,49 \times 10 = 124,9$, the Area of the $\triangle BCG$. Their Difference $104,3544 =$ the Area of the Segm. BAG .

Or the Area of any Segment may be otherwise found (as most usually it is) by a Table of the Segments of a Circle, whose Area is Unity, or 1. The Construction or making of such a Table is very well laid down in Mr. *Dariz's* Book of Gauging, Chap. 9. which he performs in this Problem.

P R O B L E M.

In a Circle whose Area is Unity, and its Diameter cut by Chord Lines into 1000 equal Parts. to find the Segment to any versed Sine propos'd, not exceeding 500 of those equal Parts.

1. Multiply the *versed Sine* propos'd by 0,002, and subtract the Product from an Unit or 1.

2. This Remainder you shall seek in the common Table of Natural Sines, (the Arch being divided into Degrees and Centesimals) which being found, let its Co-arch be doubled, and called A .

3. You must find the correspondent Sine to A ; which Sine being found, you may call S , and then it holds $6,2831853) 0,0174532925 A - S (=$ the Segment required.

Now this *Segment* being thus found, if you subduct it from an *Unit*, you have the *Co-segment*, &c.

Note, Notwithstanding what has been said in the second Precept of this *Problem*, it very often falls out that the Remainder there spoken of cannot be truly found in the Table of Natural Sines; therefore in this Case my Advice is, that you make two Operations, one with a Sine the next greater, and one with a Sine the next less; and in so doing you will be sure to have the Segment requir'd bounded between the Results of those two Operations.

Example, Let it be propos'd to find the correspondent Segment to the versed Sine 263.

First, $263 \times 0,002 = 0,526$, and $1 - 0,526 = 0,474$, its *Arch* is $28,29^\circ$ being less than just; its Complement is $61,71^\circ$, which, being doubled, is $123,42 = A$.

Then $0,0174533 A = 2,154086286$
 $= 0,8346556 = S$ the Sine of *A*.

6,2831853) 1,319430686 (0,209993 the Segment.

Now I make a second Work.

263 being multiplied with 0,002 is 526. and $1 - 526 = 0,474$ its *Arch* is $28,30^\circ$ being greater than just; and its Complement is $61,70^\circ$, which being doubled is $123,4 = A$.

Then $0,0174533 A = 2,1537372$
 $= 0,8348478 = S$ the Sine of *A*

6,2831853) 1,3188894 (0,209907 the Segment.

So you see by these two Operations that the Segment is bounded, and 'tis very probable it may be 0,20995.

But to abbreviate this large Factor, and this large Divisor, I shall here insert two Tables of them, which will be ready for Use, and exact enough too.

Divisor.		Factor.	
6,2832	1	0,0174533	1
12,5664	2	0,0349066	2
18,8496	3	0,0523599	3
25,1328	4	0,0698132	4
31,4160	5	0,0872665	5
37,6992	6	0,1047197	6
43,9824	7	0,1221730	7
50,2656	8	0,1396263	8
56,5488	9	0,1570796	9

Thus far Mr. Darie, which I have here inserted to shew the Learner how, by the Help of these two Tables, and a Table of Natural Sines, he may easily make a Table of Segments, whose Use shall be shewed farther on, viz. when I come to treat of practical Gauging. In the mean Time I shall here lay down another Method to find the Area of any Segment of a Cir-

cle (very near) by a new *Theorem*, without the Help either of a Table of *Sines* or *Segments*, having the same *Data* as before in Page 404.

Viz. Let $\begin{cases} R = \text{the Radius, or } \frac{1}{2} \text{ Diameter of the given Circle.} \\ d = \text{the Difference between the versed Sine and Radius.} \\ C = \text{half the Chord of the Segment's Base.} \end{cases}$

THEOREM. $\left\{ \frac{2\frac{1}{3}RR - 1\frac{1}{3}Rd - dd}{1\frac{1}{2}R + d} \times C = S, \text{ the Area of the Segm.} \right.$

Example, Suppose $R = BC = 16$, $d = FC = 10$, and $C = BF = 12,49$; as before.

Then $2\frac{1}{3}RR = 597,3333$. $1\frac{1}{3}Rd = 213,3333$. $dd = 100$
 $- 313,3333 = 1\frac{1}{2}Rd + dd$

$1\frac{1}{2}R + d = 34\ 284,0000$ (8,3529. Lastly, $8,3529 \times 1249 = 104,3276$ the Area of the Segment BAG , as before.

THEOREM VII.

As Squares are to the Area's of their inscribed Circles, so are Parallelograms to the Area's of their inscribed Ellipses.

That is, $\left\{ \begin{array}{l} \text{As the Square of the Diameter of any Circle : is to its} \\ \text{Area : : so is the Rectangle of the Transverse and Con-} \\ \text{jugate Diameters of any Ellipsis : to its Area.} \end{array} \right.$

Demonstration.

Circumscribe any Ellipsis with a Circle; and suppose an infinite Number of Chord Lines drawn therein, all parallel to the Conjugate Diameter, as those in the annexed Figure; then it will be $\left\{ \begin{array}{l} \text{As } (DA) \text{ the Diameter of the Circle : is to } (Nn) \text{ the Con-} \\ \text{jugate Diameter of the Ellipsis : : so is } (Bab) \text{ any Chord in} \\ \text{the Circle : to } (bab) \text{ its respective Ordinate in the Ellipsis.} \end{array} \right.$

For according to the Property of the Circle

it is 1 $TS - Ta \times Ta = \square Ba$

And by the Property of the Ellipsis

it is 2 $\square TC : \square NC :: TS - Ta \times Ta : \square ba$

1, 2 3 $\square TC : \square NC :: \square BA : \square ba$

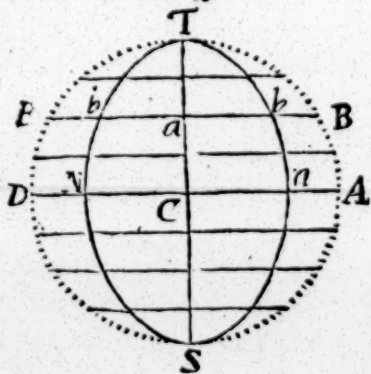
3, Hence 4 $TC : NC :: Ba : ba$

Conseq 5 $2 TC : 2 NC :: 2 Ba : 2 ba$

That is 6 $DA : Nn :: BAB : bab$

Put 7 $D = 2 TC$, and $d = 2 NC$

Then 8 $D : d :: \text{Chord } Bab : \text{Ordinate } bab, \&c.$



But

But the Sum of an infinite Series of such Chords, as BaB , do constitute the Area of the Circle, by Definition 3 : and the Sum of the like Series of their respective Ordinates, as $ba b$, do constitute the Ellipsis's Area by Definition 4. Therefore $D : d :: \text{Circle's Area} : \text{Ellipsis's Area}$, by Lemma 6. But $D : d :: DD : Dd$. Whence it follows, that $DD : \text{Circle's Area} :: Dd : \text{Ellipsis's Area}$. Q. E. D. Consequently, as 1 : is to 0,7854 :: so is the Rectangle or Product of the Transverse and Conjugate Diameters of any Ellipsis : to its Area.

Example, Suppose $TS = 36$. and $Nn = 16$; then $36 \times 16 = 576$, and $576 \times 0,7854 = 452,3904$ the Area of the Ellipsis.

Corollaries.

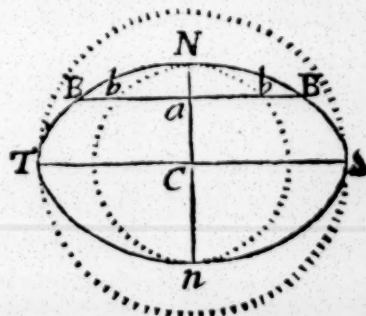
1. Hence it is easy to conceive, that the square Root of the Rectangle or Product of the Transverse and Conjugate Diameters will be the Diameter of a Circle whose Area will be equal to the Ellipsis's Area, viz. $\sqrt{576} = 24$ the Diameter of a Circle = to the Ellipsis.

2. All Segments of an Ellipsis and its circumscribing Circle (whose Bases are parallel to the Conjugate Diameter, and of the same Height) are in Proportion one to another, as their Bases are. That is, $Bab : bab :: \text{Area Segment } BNB : \text{Area Segment } bNb$; or $TS : Nn :: \text{Area Segment } BNB : \text{Area Segment } bNb$.

THEOREM VIII.

The Area of every Ellipsis is a mean Proportional between the Area's of its circumscribing and inscrib'd Circles.

The Truth of this Theorem may be easily deduced from the last; for supposing $D = TS$, and $d = Nn$, as before; then it is already proved, that $DD : Dd :: \text{circumscribing Circle's Area} : \text{Ellipsis's Area}$. But $DD : Dd :: Dd : dd$. Therefore $\text{Ellipsis's Area} : \text{inscrib'd Circle's Area} :: Dd : dd$. By Theorem 6.



Example, Let $TS = D = 36$, and $Nn = d = 16$, as before; then $DD = 1296$, and $dd = 256$.

Then

Then will $\begin{cases} 1296 \times 0,7854 = 1017,8784 \text{ the great Circle's Area} \\ 256 \times 0,7854 = 201,0624 \text{ the lesser Circle's Area.} \end{cases}$

Suppose A = the *Ellipsis's Area*; then, according to the *Theorem*, it will be, $1017,8784 : A :: A : 201,0624$. Ergo $A A = 1017,8784 \times : 201,0624 = 204657,07401216$. Consequently, $\sqrt{204657,07401216} = 452,3904 = A$, the *Area of the Ellipsis* as before in the last *Example*.

Corollary.

From hence it follows, that all *Segments* of an *Ellipsis* and its *in-scrib'd Circle*, whose *Bases* are parallel to the *Transverse Diameter*, and have the same *Height*, are in *Proportion* one to another as the *Area's* of the *Ellipsis* and *Circle* are. That is, *Area of Circle* : *Area of Ellipsis* :: *Segment b N b* : *Segment B N B*. Or, $N n : T S :: \text{Area Segment } b N b : \text{Area Segment } B N B$.

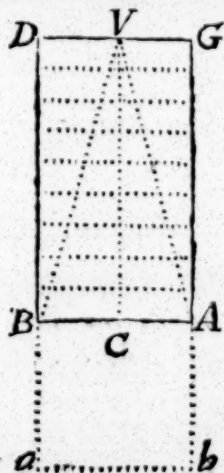
THEOREM IX.

The *Solid Content* of any *Prism* (what *Figure* soever its *Base* is of) is obtained by multiplying the *Area* of its *Base* into its *Height*.

For Instance, a *Parallelopipedon* (or *square Prism*) is constituted of an *infinite Series* of equal *Squares*; that of its *Base* BA being one of the *Terms*, and its *Height* DB , or GA , the *Number* of all the *Terms*. Consequently, the *Area* of $BAba \times DB =$ the *Sum* of all the *Series* (by *Lemma 1.*) which is the *Solidity* of the *Parallelopipedon* $DBG A$, by *Definition 10.*

Example, Suppose the *Side* of the *Base* $BA = 16$ and the *Height* $DB = 42$; then will $16 \times 16 = 256$ be the *Area* of the *Base*, and $256 \times 42 = 10752$ the *Solid Content* of the *Parallelopipedon* $DBG A$.

In this Manner you may find the *Solidity* of all regular *Polygonous Prisms*, whose *Bases* (or *Ends*) are parallel and alike, what *Form* soever they are of, that is, whether their *Bases* are *Triangles*, *Pentagons*, *Hexagons*, or *Octagons*, &c.



THEO-

THEOREM X.

Every Pyramid is the third Part of the Prism, that hath the same Base and Height with it (7. e. 12.)

That is, the Solid Content of the Pyramid BVA (in the last Figure) is one Third of its circumscribing Prism $DBG A$.

Demonstration.

For every Pyramid that hath a square Base (as $B A b a$, in the last Figure) is constituted of an infinite Series of Squares, whose Sides or Roots are continually increasing in Arithmetick Progression, beginning at the Vertex or Point V (See Theor. 2.) its Base $B A B a$; being the greatest Term ($= L L$) and its perpendicular Height VC , or DB , is the Number of all the Terms $= N$; but $\frac{NLL}{3} = S$ the Sum of all the Series, by Lemma 3, and $S =$ the Solid Content of the Pyramid BVA , by Definition 13.

Example, Suppose the Side of a Pyramid's Base be $BA = 16$. and its Height be $VC = 42$. Then $16 \times 16 = 256$ the Area of its Base $B A B a = a$, and $\frac{256 \times 42}{3} = 3584$. Or $\frac{356}{3} \times 42 = 3584$ or thus, $256 \times \frac{42}{3} = 3584$, is the Solidity of that Pyramid BVA .

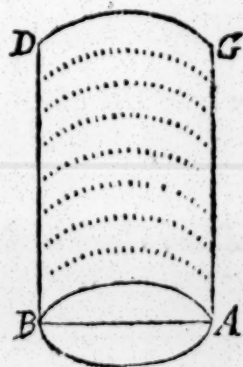
Corollary.

From hence it will be easy to conceive, that every Pyramid is $\frac{1}{3}$ of its circumscribing Prism, what Form soever its Base is of, viz. whether it be a Square, Triangle, Pentagon, &c.

THEOREM XI.

The Solid Content of every Cylinder is obtain'd by multiplying the Area of its Base into its Height.

For every Right Cylinder is only a round Prism, being constituted of an infinite Series of equal Circles; that of its Base or End being one of the Terms, and its Height BD is the Number of all the Terms. Therefore the Area of its Base BA , being multiply'd into DB , will be its Solidity, by Lemma 1. viz. Let $D = BA$, and $H = GA$. Then $DD \times H =$ its Solidity.



Example,

Example, Let the *Diameter* of its *Base* be $D = 16$, and its *Height* $H = 42$. Then $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$ the *Area* of its *Base*. And $201,0624 \times 42 = 8444,6208$ the *Solid Content* of that *Cylinder D B G A*.

Corollary.

Hence it is evident that every square *Parallelopipedon* is to its *in-scrib'd Cylinder*, as $1 : 0,7854$. Or in whole *Numbers*, as $452 : 355$ very near. And that all *Prisms* are in *Proportion* to their *in-scrib'd Cylinders*, as the *Area's* of their *Bases* are.

T H E O R E M XII.

The Curve Superficies of every Right Cylinder is equal to the Rectangle made of its Height into the Periphery of its Base.

That is, DB multiply'd into the *Periphery* of the *Diameter B A*, will produce the *Curve Superficies* of the last *Cylinder D B G A*. For the *Cylinder* is constituted of an *infinite Series* of equal *Circles* (according to the last *Theorem*.) Therefore its *Curve Superficies* is compos'd of the *Peripheries* of those *Circles*, by *Definition* 20. But the *Periphery* of its *Base B A* is one of the *Terms*, and its *Height D B* is the *Number of Terms*. Therefore, &c. as by *Lemma 1*. To which, if there be added the *Area's* of both its *Ends* (or *Bases*) the *Sum* will be the *Superficies* of the whole *Cylinder*.

Example. Suppose the *Diameter* of its *Base* to be $BA = 16$. and its *Height D B* $= 42$; as before, then $1 : 3,1416 :: 16 : 50,2656$ the *Periphery* of its *Base*. Again, $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$ the *Area* of each *End* or *Base*.

Then $50,2656 \times 42 = 2111,1552$ the *Curve Superficies*, to which add $201,0624 \times 2 = 402,1248$ both the *End Area's*.

The *Sum* $= 2513,2800$ is the *Superficies* of the whole *Cylinder*.

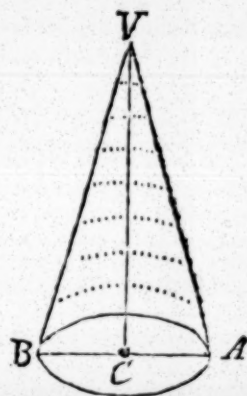
T H E O R E M XIII.

Every Cone is the third Part of a Cylinder, having the same Base with it, and their Altitudes equal. (10. e. 12.)

Demon-

Demonstration.

The Truth of this *Theorem* may be easily conceiv'd by only considering, that a *Cone* is but a round *Pyramid*, and therefore it must needs have the same *Ratio* to its circumscribing *Cylinder* as the square *Pyramid* hath to its circumscribing *Parallelopipedon*, viz. as 1 : to 3. However, to make it yet clearer, let it be farther considered, that every *Right Cone* is constituted of an infinite *Series* of *Circles*, whose *Diameters* do continually encrease in *Arithmetick Progression* beginning at the *Vertex* or *Point V*, the *Area* of its *Base B A* being the greatest *Term*, and its perpendicular *Height V C* the *Number* of all the *Terms*; therefore the *Area* of the Circle $B A \times \frac{1}{3} V C$ will be the *Sum* of all the *Series*, by *Lemma 3*, which is the *Cone's Solidity*.



Example. Let the *Diameter* of its *Base* be $B A = 16$, and its *Height* $V C = 42$; Then $1 : 0,7854 :: 16 \times 16 = 256 : 201,0624$ the *Area* of the *Base*; and $\frac{201,0624 \times 42}{3} = 2814,8736$ the *Solidity* of the *Cone B V A*. Or thus, $201,0624 \times \frac{42}{3} = 2814,8736$, &c.

Corollary.

Hence it follows, that every square *Pyramid* is to its inscrib'd *Cone*, as 1 : 0,7854. (Or as 452 : 355) consequently, that all *Pyramids* have the same *Ratio* to their inscrib'd *Cones* as the *Area's* of their *Bases* have.

T H E O R E M XIV.

The Curve Superficies of every Right Cone is equal to half the Rectangle of the Periphery of its Base into the Length of its Side.

The Truth of this *Theorem* is self-evident from the *Definition* of a *Cone*, Chap. 1, Part IV, where it appears, that the *Curve Superficies* of every *Right Cone* (as $B V A$) is equal to the *Area* of a *Sector* of that Circle whose *Radius* is the *Side* of the *Cone* ($V B$) and its *Arch* equal to the *Periphery* of the *Cone's Base* ($B A$). But the *Area* of any *Sector* is equal to half the *Rectangle* of the *Radius* into its *Arch*, by *Theorem 4*. Therefore, &c.

H h h

Exam-

Example. Suppose the Length of the Cone's Side to be VB , or $VA = 42,7551$, and the Diameter of its Base, viz. $BA = 16$ (as before) then will $50,2656$ be the *Periphery* of its Base, and $\frac{50,2656 \times 42,7551}{2} = 1074,5553$, &c. the *Curve* of the *Superficies*; to which if there be added the *Area* of its Base, the Sum will be the *Superficies* of the whole (viz. *all the*) Cone.

That is, $1074,5553$
 $+ 201,0624$ the *Area* of the Base.

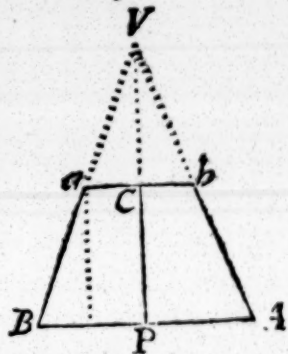
Sum $1275,6177$ is the total *Superficies*, &c.

Note, *The Truth of this Theorem may be prov'd from the Consideration of the last Theorem and Definition 20.*

Scholium.

From the 10th and 13th *Theorems* may be easily deduced several *Theorems* for finding the *solid Content* of any *Frustrum* or *Part*, either of a *Pyramid* or *Cone*, cut by a plain Parallel to its *Base*.

Suppose a square *Pyramid*, as BVA , to be cut by a *Plain* at ab , parallel to its *Base* BA , and it were requir'd to find the *Solidity* of the *Frustrum* or *Part* $abAB$; let there be given $D = BA$ the Side of the greater Base. $d = ba$ the Side of the lesser Base. $H = CP$ the perpendicular Height.



First,	1	$D - d : H :: d : \frac{dH}{D-d} = VC$ by the Figure.
Then	2	$\left\{ DD \times \frac{H+VC}{3} = \text{the whole Pyramid } BVA. \right.$ By Theorem 10.
And	3	$dd \times \frac{1}{3} VC = \text{the Pyramid } aVb \text{ cut off.}$
Viz. 1, 2	4	$\left\{ \frac{DDDH}{3D-3d} = \text{the whole Pyramid } BVA. \right.$
And 1, 3	5	$\left\{ \frac{dddH}{3D-3d} = \text{the Pyramid } aVb. \right.$
4 — 5	6	$\left\{ \frac{DDDH - dddH}{3D-3d} = \text{the Frustrum } abAB. \right.$
6. Reduc.	7	$\frac{DD+Dd+dd}{3} \times \frac{1}{3} H = \text{the Frustrum } abAB.$

Which in Words gives this following *Theorem*.

T H E O.

THEOREM XV.

To the Rectangle of the Sides of the two Bases, add the Sum of their Squares; that Sum, being multiply'd into one Third of the Fruustum's Height, will give its Solidity.

Example. Suppose the Side of the greater Base $BA = 16$, and the Side of the lesser Base (or Top) $ab = 12$ the Height $CP = 9$. Then $16 \times 12 = 192$. $16 \times 16 = 256$. and $12 \times 12 = 144$. Next $192 + 256 + 144 = 592$. and $\frac{592 \times 9}{3} = 1776$. Or $592 \times 3 = 1776$ the Content of the Fruustum of a square Pyramid.

And if it were the like Fruustum of a Right Cone, it may be found by the same Theorem. Supposing $D =$ the Diameter of the greater Base, $d =$ the Diameter of the lesser, and $H =$ the Height of the Fruustum, then the Sum of all the Squares which constitute the Fruustum of a square Pyramid, are to the Sum of all the Circles which constitute the like Fruustum of a right Cone, in the Ratio of 1 : to 0,7854 (or of 452 : to 355) therefore it will be $1 : 0,7854 :: \overline{DD + Dd + dd} \times \frac{1}{3} H : 0,7854 \overline{DD + Dd + dd} \times \frac{1}{3} H =$ the Cone's Fruustum, that is, in the last Example, $1 : 0,7854 :: 1776 : 1394,8704$ the like Fruustum of a right Cone. Or because $\frac{1}{0,7854} = 1,273236$, &c. Therefore it may be made $1,273236 \overline{DD + Dd + dd} \times \frac{1}{3} H (=$ the same Fruustum; that is, $1,273236) 1776 (1394,87, \&c.$ as before. And if you take the Triple of this Divisor, viz. $1,273236 \times 3$, it will be $3,8197 \overline{DD + Dd + dd} : \times H (=$ the Fruustum, &c.

Again,

Suppose 1 $x = D - d$, and $F =$ the Fruustum.

Then 2 $\overline{DD + Dd + dd} = \frac{3F}{H}$, by the 7th Step of the last.

1 $\odot^2$ 3 $xx = DD - 2Dd + dd$

2 — 3 4 $3Dd = \frac{3F}{H} - xx$

4 $\div$ 3 5 $Dd = \frac{F}{H} - \frac{1}{3}xx$, or $Dd + \frac{1}{3}xx = \frac{F}{H}$

5 $\times H$ 6 $\overline{Dd + \frac{1}{3}xx} \times H = F$ the Fruustum $abAB$.

Hence we have another easy Theorem for finding the same Fruustum.

H h h 2

THEO-

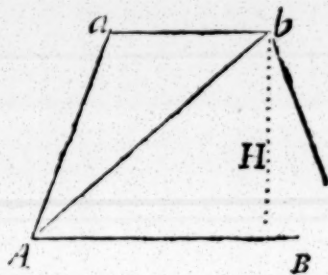
THEOREM XVI.

To the Rectangle of the Sides of the two Bases, add one third Part of the Square of their Difference; that Sum, being multiply'd into the Height, will produce the Solidity.

Example. Let $D = 16$. $d = 12$. and $H = 9$, as before; then $Dd = 192$. $D - d = 4 = x$. $\frac{1}{3}xx = \frac{4 \times 4}{3} = 5,3333$, and $192 + 5,3333 = 197,3333$. Lastly, $197,3333 \times 9 = 1775,9997$ the Solidity of the *Frustum* of the square Pyramid, as before. And 3,81968) 1775,9997 (1394,87, &c. the like *Frustum* of a right Cone, as before.

Either of the two last *Theorems* (being rightly apply'd) will produce the true *solid Content* of all *Frustums* of any kind of Pyramids, that are intercepted between two parallel and alike *Plains* or *Bases*: As above.

But if such *Frustums* are cut through the Extremities of both Bases by a Diagonal Plain (as Aab in the annexed Figure) into two Parts, Aab and ABb , call'd *Hoofs*; then the Solidity of those *Hoofs* is usually found by dividing the middle Term Dd of the Equation $DD + Dd + dd$ into two Parts, and adding one of those Parts to the Square of each Base. Thus, $\overline{DD + \frac{1}{2}Dd} \times \frac{1}{3}H =$ the great Hoof ABb , and $\overline{dd + \frac{1}{2}Dd} \times \frac{1}{3}H =$ the lesser Hoof Aab of the *Frustum* of any square Pyramid. Then 3,8197) $\overline{DD + \frac{1}{2}Dd} \times H (=)$ the greater Hoof of a Cone. And 3,8197) $\overline{dd + \frac{1}{2}Dd} \times H (=)$ the lesser Hoof, &c.



These are the *Theorems* made Use of by Mr. *Daric*, in his Book of *Gauging*, and are pretty near the Truth, but not exactly so; for they give the Solidity of the upper Hoof Aab a small Matter too big, and the lower Hoof ABb as much too little.

Now, in order to rectify that small Error, I shall here propose the two following *Theorems*, which come very near the Truth, and are more easily perform'd than those propos'd in the first Impression of this Book.

First, $\overline{DD + \frac{1}{2}Dd + D - d} \times \frac{1}{3}H$ will be the Solidity of the greater Hoof ABb .

Secondly,

Secondly, $dd + \frac{1}{2} Dd + d - D \times \frac{1}{3} H$ will give the Solidity of the lesser Hoof Aab , of the Frustum of any square Pyramid.

And for the like Hoofs of the Frustum of any right Cone, it will be

Thus, 3,8197) $\frac{DD + \frac{1}{2} Dd + D - d \times H}{dd + \frac{1}{2} Dd + d - D \times H}$ (= the greater Hoof.
And 3,8197) $\frac{DD + \frac{1}{2} Dd + D - d \times H}{dd + \frac{1}{2} Dd + d - D \times H}$ (= the lesser Hoof.

Note, In order to avoid many Words in the following Demonstrations, let $\odot$ signify any Circle in general; and if any two Letters be join'd to it, thus, $\odot BA$, &c. it then denotes the Area of such a Circle as those two Letters represent the Radius of.

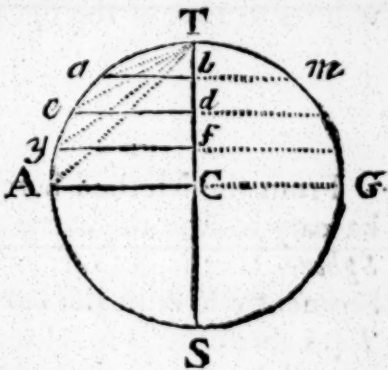
THEOREM XVII.

The Superficies of every Sphere (or Globe) is equal to four Times the Area of its greatest Circle.

That is, of a Circle whose Diameter is the Axis of the Sphere.

Demonstration.

If any Semicircle (as $ATGS$) be turn'd or mov'd about its Diameter (TS) it will describe a solid Body call'd a Sphere, which will be constituted of an infinite Series of concentrick or parallel Circles, whose Diameters are Chords, viz. $\odot ab$, $\odot ed$, $\odot ef$, &c. by Definition 14. Consequently, the Superficies of the Sphere will be compos'd of the Peripheries of those Circles which constitute its Solidity, by Definition 20.



Let $D = TS$, the Axis of any Sphere. Then, according to the Property of a Circle, it

will be	1	$D - Tb \times Tb = \square ab$
That is,	2	$D \times Tb - \square Tb = \square ab$
Therefore	3	$D \times Tb = \square aT$, for $\square ab + \square Tb = \square aT$.
And {	4	$D \times Td = \square eT$
	5	$D \times Tf = \square yT$, &c.

* The Error is here corrected, which Mr. J. Robertson takes Notice of in his Book, entitled, *A Compleat Treatise of Mensuration*, Page 160.

Hence

Hence 'tis evident, that the *Series* $\square a T$, $\square e T$, $\square y T$, &c. are in the same *Ratio* with $T b$, $T d$, $T f$, &c. viz. in *Arithmetick Progression*; whence it follows, that the $\odot a T$ = the Sum of all the Circle's *Peripheries* between T and b , and $\odot e T$ = the Sum of all the Circle's *Peripheries* between T and d , &c. Consequently, that the $\odot A T$ = the Sum of all the Circle's *Peripheries* included between T and C ; that is, $\odot A T$ = the *Superficies* of the *Hemisphere*. And because $\square A C + \square T C = \square A T$, and $\square A C = \square T C$. Therefore $\odot A T = 2 \odot A C$ is the *Superficies* of the *Hemisphere*. Consequently, $4 \odot A C$ will be the *Superficies* of the whole *Sphere*. Q. E. D.

Example. Suppose the Axis $T S = D = 16$. Then $D D = 256$, and $1 : 0,7854 :: 256 : 201,0624 = \odot A C$, for $\frac{1}{2} D = A C$. Then $201,0624 \times 4 = 804,2496$, the *Superficies* of the whole *Sphere*. Or, because $3,1416$ is four Times $0,7854$, therefore it will always be $1 : 3,1416 :: D D : 3,1416 D D$, the *Superficies* of the *Sphere* (as before); and it is equal to the *curve Superficies* of the *right Cylinder*, whose Diameter and Height are each = D the Axis of the *Sphere*. For $3,1416 D$ = the *Periphery* of the *Cylinder's* Base, and that, multiply'd with D its Height, will be $3,1416 D D$ the *curve Superficies* of the *Cylinder*, by *Theorem 12*. And if to this there be added the *Area* of its two Bases (or *Ends*) viz. $1,5708 D D$, then 'tis evident, that the whole *Superficies* of the *Cylinder* will be to that of the *Sphere* in Proportion of 3 to 2.

Scholium.

From the Method here used in proving the last *Theorem* 'twill be easy to find the *curve Superficies* of any *Segment*, or *Part* of a *Sphere*, that is cut off by a Right Line or Plain, viz. such as the Segment $a T m$ in the last *Scheme*, whose *curve Superficies* is $\odot a T$ (as above). Therefore (because $\square a b + \square T b = \square a T$) it will be $\odot a b + \odot T b$ = the *curve Superficies* of that *Segment*.

But if the Axis $T S$, and Height $T b$, of the *Segment* are given, then will it be $T S \times T b \square a T$; as in the third Step above: Which gives this Proportion or *Theorem*;

Viz.

Viz. $\left\{ \begin{array}{l} \text{As the Axis of the Sphere : is to the whole Superficies of the} \\ \text{Sphere : : so is the Height of any Segment to its curve Su-} \\ \text{perficies.} \end{array} \right.$

To which if there be added the *Area* of the Segment's Base, the Sum will be the Superficies of the whole Segment.

THEOREM XVIII.

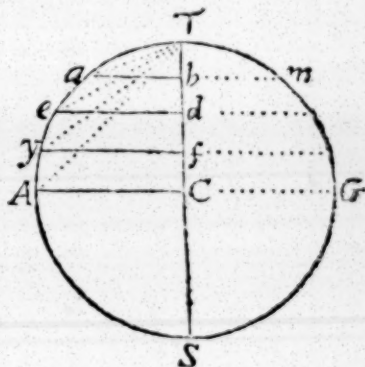
Every Sphere is equal to two Thirds of its circumscribing Cylinder.

That is, of a Cylinder whose Height and Diameter of its Base are each equal to the Axis of the Sphere.

Demonstration.

According to the Work in the last *Theorem* it appears, that $\odot ab$, $\odot ed$, $\odot yf$, &c. do constitute the Solidity of the Sphere; and that $\square aT$, $\square eT$, $\square yT$, &c. are a Series of Terms in *Arithmetick* Progression, $\square AT$ being the greatest Term, and TC the Number of Terms; therefore $\odot AT \times \frac{1}{2} TC =$ the Sum of all the Series, per *Lemma 2*. And because $\square aT - \square Tb = \square ab$, $\square eT - \square Td = \square ed$, $\square yT - \square Tf = \square yd$, $\square AT - \square TC = \square AC$, &c. wherein $\square Tb$, $\square Td$, $\square Tf$, &c. are a Series of Squares whose Roots Tb , Td , Tf , are in *Arithmetick* Progression, $\square TC$ being the greatest Term, and TC the Number of Terms; therefore $\odot TC \times \frac{1}{3} TC =$ the Sum of all that Series, per *Lemma 3*, consequently, $\odot AT \times \frac{1}{3} TC - \odot TC \times \frac{1}{3} TC =$ the Sum of the Series $\odot ab$, $\odot ed$, $\odot yf$, &c. which constitute the Solidity of the half Sphere ATG . Put $D = 2TC$ the Axis of the Sphere; then $\frac{1}{4} D = \frac{1}{2} TC$, and $\frac{1}{6} D = \frac{1}{3} TC$. And because $\square AT = 2 \square TC$; therefore $\odot AT = 2 \odot TC = 1,5708 DD$. And $1,5708 DD \times \frac{1}{4} D = 0,3927 DDD$.

Again, $D TC \times \frac{1}{3} TC = 0,7854 DD \times \frac{1}{6} D = 0,1309 DDD$, then $0,3927 DDD - 0,1309 DDD = 0,2618 DDD$ the Solidity of the *Semi-sphere* ATG ; consequently, $0,2618 DDD \times 2 = 0,5236 DDD$ will be the solid Content of the whole *Sphere*, which is equal to two Thirds of the Cylinder whose Diameter of its Base and Height $= D$. For $0,7854 DDD =$ the Solidity of the Cylinder, by *Theorem 11*. But $\frac{2}{3}$ of $0,7854 DDD = 0,5236 DDD$; as before. Therefore, &c. : as by *Theorem*.



Example. Suppose the *Axis* $D = 16$, then $DDD = 4096$, and $1 : 0,5236 :: 4096 : 2144,6656$ the *solid Content* of that *Sphere*.

Corollaries.

1. Hence it appears, that the *solid Content* of every *Sphere* is equal to its *Superficies* multiply'd into one sixth Part of its *Axis*. For its *Superficies* is $3,1416 DD$, by *Theorem 17*. But $3,1416 \times \frac{1}{6} D = 0,5236 DDD$ the *solid Content*, as before.

2. And hence 'tis also evident, that there is the like *Ratio* or *Habitude* between the *Cube* and its *inscrib'd Sphere*, as is betwixt the *Square* and its *inscrib'd Circle*; and that is, as the *Superficies* of any *Cube*: is to the *Superficies* of its *inscrib'd Sphere*: so is the *solid Content* of that *Cube*: to the *solid Content* of the *Sphere*. [See the *Circle's Proportion*, *Page 407*.] For if $D =$ the *Side* of the *Cube*, then $6 DD =$ its *Superficies*, and $DDD =$ its *Solidity*, and $3,1416 DD =$ the *Sphere's Superficies*. But $6 DD : 3,1416 DD :: DDD : 0,5236 DDD$ the *Solidity* of the *Sphere*; as above.

Scholium.

From the *Proof* of this *Theorem* 'twill be easy to deduce or raise *Theorems* for finding the *solid Content* of any *Frustrum* or *Segment* of a *Sphere*, as aTm in the last *Figure*. For we there suppose the *Segment* aTm to be constituted of an infinite *Series* of *Circles*, which have the same *Ratio* with all those *Circles* that constitute the *Semi-sphere*. Therefore it follows, that $\odot at \times \frac{1}{2} Tb - \odot bT \times \frac{1}{3} Tb$ will be the *Sum* of all the *Circles* intercepted between T and b . Consequently 'twill be the *Solidity* of that *Segment*. And because $\square ab + \square Tb = \square aT$: therefore $\odot ab + \odot Tb \times \frac{1}{2} Tb - \odot Tb \times \frac{1}{3} b =$ the same *Solidity*.

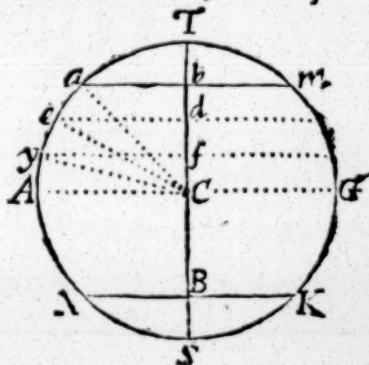
Let $c = ab$ half the *Segment's Base*; $h = Tb$ its *Height*; and $S =$ the *Solidity* of the *Segment* or *Frustrum*: Then $\odot ab = 3,1416cc$, and $\odot Tb = 3,1416hh$. Consequently,

$$\frac{3,1416cch + 3,1416hhh}{2} - \frac{3,1416hbh}{3} = S,$$
 which being reduced will become $3cch + hhh \times 0,5236 = S$. Or $1,909855) 3cch + hhh$ ($= S$. for $0,5236$) $1,0000 (1,909855$. Which is one *Theorem* for finding the *Frustrum's Solidity*.

Note,

Note, Here we suppose the Height of the Segment, and the Diameter of its Base to be given; but if the *Axis* of the Sphere, and the Height of the Segment be given, then putting $D =$ the Sphere's Axis, $b =$ the Segment's Height, and c as before, 'twill be $D - b \times b = cc$, viz. $D b - b b = cc$. Therefore $3 D b b - 2 b b b = 3 c c b + b b b$. consequently $3 D b b - 2 b b b \times 0, 5236 = S$, the *Frustrum's* Solidity. Or $1,90985) 3 D b b - 2 b b b (= S$, as before. Which is a second *Theorem* for finding the same *Frustrum a Tm*.

And if it be requir'd to find the middle Part $a m N K$, usually call'd the middle *Zone* of a Sphere, then because 'tis suppos'd that $a m = N K$, or which is all one, that $b C = C B$, therefore it is plain, that, if twice the Segment $a T m$ be taken from the Solidity of the whole Sphere, there will remain the Middle *Zone a m N K*. But, because that Work is a little troublesome, I shall here shew how to raise a *Theorem* for the doing it.



First, Because $AC = yC = eC = aC = TC$. Therefore it will be $\square AC - \square Cf = \square yf$. $\square AC - \square Cd = \square ed$. $\square AC - \square Cb = \square ab$, &c. Here because $\square AC$. $\square AC$. $\square AC$, &c. are a Series of *Equals*, and Cb the Number of all the Terms, therefore $\square AC \times Cb =$ the Sum of all that Series, by *Lemma 1*. And $\square Cf$. $\square Cd$. $\square Cb$, &c. being a Series of Squares whose *Roots* are in *Arithmetick Progression*, beginning at the Center or Point C , viz. 0 , Cf , Cd , Cb , &c. wherein the greatest Term is $\square Cb$, and Number of Terms is Cb . Ergo $\square Cb \times \frac{1}{3} Cb =$ the Sum of all the Series by *Lemma 3*. Consequently, the $\odot AC \times Cb - \odot Cb \times \frac{1}{3} Cb =$ the Sum of all the Series $\odot yf$. $\odot ed$. $\odot ab$, &c. which do constitute the Solidity of the *half Zone a m A G*. And because $\square AC - \square Cb = \square ab$. Ergo $\odot AC - \odot ab = \odot Cb$. Consequently $\odot AC \times Cb - \frac{\odot AC - \odot ab \times Cb}{3} = \frac{2 \odot AC + \odot ab \times \frac{1}{3} Cb}{3}$ will be the Solidity of the *half Zone*.

Put $D = AG = 2 AC$. $x = a m$. and $H = b B = 2 Cb$. Then $\odot AC = 0,7854 D D$. $\odot ab = 0,7854 x x$. And if we turn the common Factor $0,7854$ into the Divisor $1,27323$, and

and then take the Triple of that Divisor, viz. 3,8197 (as before in the Frustrum of Pyramids) the Result of the precedent Work will produce this following Theorem.

THEOR. XIX. $\left\{ \frac{2DD+xx}{3,8197} : \times H = \right\}$ the middle Zone am
NK.

THEOREM XX.

Spheres are in Proportion one to another as the Cubes of their Diameters.
(18. e. 12.)

Demonstration.

Suppose D = the Diameter or Axis of any Sphere, and d = the Diameter of another Sphere, either greater or lesser. Then is $0,5236 DDD$ = the Solidity of one Sphere, and $0,5236 ddd$ = the Solidity of the other Sphere, by Theorem 18. But $DDD : ddd :: 0,5236 DDD : 0,5236 ddd$. Q. E. D.

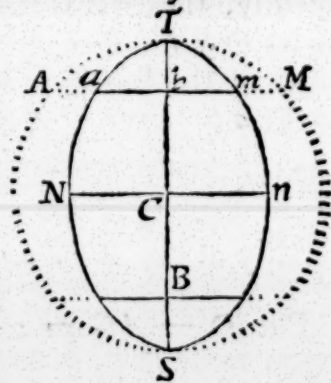
THEOREM XXI.

The solid Content of every Spheroid is equal to two Thirds of its circumscribing Cylinder.

Demonstration.

Suppose the Figure $NTnSN$ in the annex'd Scheme, to represent a Spheroid, form'd by the Rotation of the Semi-Ellipsis TNS , about its Transverse Axis TS (as by Definition 15.)

Let $D = TS$, the Length of the Spheroid, and the Axis of its circumscribing Sphere; and $d = Nn$, the Diameter of the greatest Circle of the Spheroid. Then because $\square TC : \square NC :: \square Ab : \square ab$, by Step 3 in Theor. 7, therefore it will be $DD : dd :: \square Ab : \square ab :: \odot Ab : \odot ab$, &c. But the Sum of an infinite Series of such Circles as $\odot Ab$ (whose Diameters are Chords) do constitute the Solidity of the Sphere, (as before at Theorem 18) and the Sum of an infinite Series of such Circles as $\odot ab$ (viz. whose Diameters are Ordinates of the Ellipsis) do constitute the Solidity of the Spheroid, by Definition 15. Ergo $DD : dd :: 0,5236 DDD : 0,5236 dd$ = the Solidity of the Spheroid, by Lemma 6.



But

But $0,5236dd D = \frac{2}{3}$ of the Cylinder whose Diameter is $= d$, and Height $= D$, by Theorem 11. Q. E. D.

Now, from this Proportion between the Sphere and its inscrib'd Spheroid, 'twill be very easy to deduce Theorems for finding the Solid Content either of the Segment or middle Zone of any Spheroid, having the same Height with that of the Sphere.

For { As the Solidity of the whole Sphere : is to the Solidity of the whole Spheroid : : so is any Part of the Sphere : to the like Part of the Spheroid, by the Converse to Lemma 6.

As for Instance ; suppose it were requir'd to find the middle Zone of any Spheroid : Let $D = TS$, and $d = Nn$, as above ; and $H = bB$, $x = AM$, as in Theorem 19, and let $c = am$. Then

$\frac{2DD + xx}{3,8197} \times H =$ the middle Zone of the Sphere. And $0,5236 DDD$
 $: 0,5236 dd D :: \frac{2DD + xx}{3,8197} \times H : \frac{2dd \times H}{3,8197} + \frac{xx dd \times H}{3,8197 DD} =$ the middle Zone of the Spheroid.

Again, $DD : dd :: xx : cc$, therefore $\frac{xx dd}{DD} = cc$. consequently,
 $\frac{xx dd}{DD} \times \frac{H}{3,8197} = \frac{cc}{3,8197} \times H$, which being taken instead of $\frac{xx dd \times H}{3,8197 DD}$, there will arise this following

THEOREM XXII. { $\frac{2dd + cc}{3,8197} : \times H =$ { the middle Zone of the Spheroid
 being the very same with Theorem 19.

Note, In the same Manner you may raise Theorems for finding the Segment of a Spheroid, cut off at either of its Ends, &c.

THEOREM XXIII.

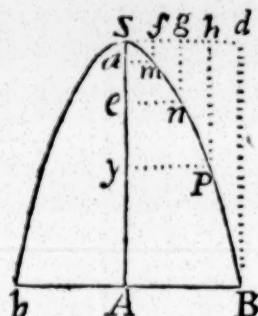
The Area of every Parabola is equal to two Thirds of its circumscribing Parallelogram.

Demonstration.

Let the Figure SAB represent half a Parabola. Make DB parallel to the Axis SA , and Sd parallel to the Semi-Ordinate AB , and suppose Sd to be divided into an infinite Series of equidistant Points,

Points, as f, g, h , &c. and from those Points imagine a Series of parallel Lines, viz. $f m, g n, h p$, &c. to touch the Curve of the Parabola, and meet the Semiordinates $m a, n e, y p$, &c. Then, according to the Property of the Parabola, it will

be {	1	$SA : \square AB :: Sa : \square am$
	2	$SA : \square AB :: Se : \square en$
	3	$SA : \square AB :: Sy : \square yp$, &c.
But		$Sa = fm, Se = gn, Sy = hp, SA = dB$
		Therefore alternately it will be
3,	4	$\square AB : dB :: \square yp : hp$
2,	5	$\square AB : dB :: \square en : gn$
1,	6	$\square AB : dB :: \square am : fm$, &c.



In these Proportions $\square am, \square en, \square yp$, &c. are a Series of Squares whose Roots Sf, Sg, Sh , &c. are in Arithmetick Progression, beginning at the Point S . And because the Lines hp, gn, fm , &c. have the same Ratio, therefore they are as such a Series of Squares, wherein dB is the greatest Term, and Sd the Number of Terms.

Consequently $\frac{dB \times Sd}{3} =$ the Sum of all those Lines, by Lemma 3.

But $SA \times AB = dB \times Sd$. Therefore $\frac{SA \times AB}{3} =$ the Sum of all that Series of Lines; but all those Lines do constitute the Area of the Semi-Parabola's Complement, viz. the Area of what half the Parabola wants of completing or filling up the Parallelogram $SdAB$.

Wherefore $SA \times AB - \frac{1}{3} SA \times AB = \frac{2}{3} SA \times AB$ will be the Area of half the Parabola SAB . Consequently, $\frac{2}{3} SA \times bB$ will be the Area of the whole Parabola bSB . Q. E. D.

Example. Suppose the Base, or greatest Ordinate, of a Parabola to be $bB = 24$, and its intercepted Diameter (or Axis) be $SA = 33$; then $2 SA \times bB = 66 \times 24 = 1584$. and $\frac{2}{3} 1584$ (528 the Area of that Parabola.

THEOREM XXIV.

Every Parabolick Conoid is equal to one Half of its circumscribing Cylinder.

Demon

Demonstration.

If any *Semi-Parabola* (as $B S A$) be turn'd or mov'd about its *Axis* ($S A$) 'twill form a *solid Parabolical Conoid*, constituted of an *infinite Series of Circles*, viz. $\odot b a$, $\odot f e$, $\odot g y$, &c. by *Definition 17*.

Now, according to the *Property* of every *Parabola*, it will be,
 $S A : A B :: A B : \frac{\square A B}{S A} = L$, the *Latus Rectum*.

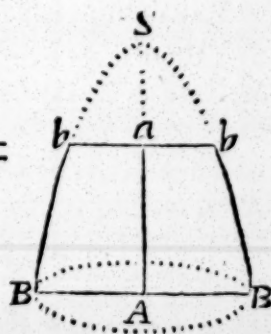
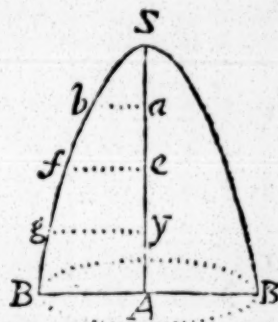
$$\text{Then } \begin{cases} S a \times L = \square b a \\ S e \times L = \square f e \\ S y \times L = \square g y, \text{ \&c.} \end{cases}$$

Here $S a \times L$, $S e \times L$, $S y \times L$, &c. are a *Series of Terms* in *Arithmetick Progression*: therefore $\square b a$, $\square f e$, $\square g y$, &c. are also a *Series of Terms* in the same *Progression*, beginning at the Point S ; wherein $\square A B$ is the greatest *Term*, and $S A$ the *Number* of all the *Terms*. Therefore $\square A B \times \frac{1}{2} S A =$ the *Sum* of all the *Series* by *Lemma 2*. Consequently, $\odot A B \times \frac{1}{2} S A =$ the *Sum* of all the *Series* $\odot b a$, $\odot f e$, $\odot g y$, &c. which do constitute the *Solidity* of the *Conoid*. And putting $D = 2 A B$, and $H = S A$. Then $0,7854 D D \times \frac{1}{2} H = 0,3927 D D H$ will be the *solid Content* of the *Conoid*, which is just half the *Cylinder* whose *Base* $= D$ and *Height* $= H$. [See *Theorem 11*.] Q. E. D.

This being understood, 'twill be easy to raise a *Theorem* for finding the lower *Frustum* of any *Parabolick Conoid*. For supposing $h = a A$ the *Height* of the *Frustum*, and $p = S a$ the *Height* of the Part $b S b$ cut off; then $h + p = S A$, the *Height* of the whole *Conoid*. Consequently, $\frac{\odot A B \times h + \odot A B \times p}{2} =$ *Solidity* of

the whole *Conoid*. And $\frac{b a + p}{2} =$ the *Solidity* of the Part cut off.

$$\begin{array}{lcl} \text{Ergo} & 1 & \left\{ \frac{\odot A B \times h + \odot A B \times p - \odot b a \times p}{2} = \right. \\ & & \left. \text{the Solidity of the Frustum.} \right. \\ \text{But} & 2 & h + p : \square A B :: p : \square b a \\ \text{Conseq.} & 3 & h + p : \odot A B :: p : \odot b a \\ 3 & \therefore 4 & \odot A B \times p = \odot b a \times h + \odot b a \times p \end{array}$$



$$\begin{array}{r|l}
 4 - \odot ba \times p & 5 \mid \odot AB \times p - \odot ba \times p = \odot ba \times b \\
 1 \times 2 & 6 \mid \odot AB \times b + \odot AB \times p - \odot ba \times p = 2 F \\
 6 - 5 & 7 \mid \odot AB \times b = 2 F - \odot ba \times b \\
 7 + \odot ba \times b & 8 \mid \odot AB \times b + \odot ba \times b = 2 F \\
 8 \div 2 & 9 \mid \frac{\odot AB + \odot ba}{2} \times b = F \text{ the Frustum's Solidity.}
 \end{array}$$

Let $D = AB$, as before, and $d = 2ba$ the Diameter of the Part cut off; then we shall have this following

THEOREM XXV. $\left\{ \begin{array}{l} 0,3927 DD + 0,3927 dd \times b = \text{the} \\ \text{Solidity of the Frustum requir'd.} \end{array} \right.$

Or $\left\{ \frac{DD + dd}{2,5464} \times b = \text{the Frustum; for } ,3927) 1,0000 (= 2,5464 \right.$

and because $2,5464 + \frac{2,5464}{2} = 3,8196$; therefore it may be

made $3,8196) DD + dd \times \frac{1}{2} b (= \text{the same Frustum, \&c.}$

Note, *The Reason why I have reduced this Theorem to have the same Divisor with those at the Frustums of Pyramids, \&c. will best appear farther on, viz. when they all come to be apply'd to Practice in Gauging.*

THEOREM XXVI.

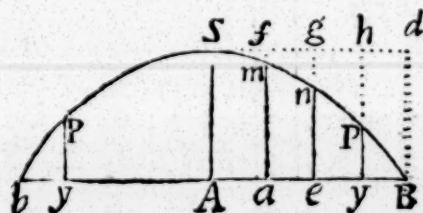
Every Parabolick Spindle (or Pyramidoid) is equal to eight Fifteenths of its circumscribing Cylinder.

Demonstration.

If any acute Parabola, as bSB , be turn'd or mov'd about its greatest Ordinate bAB , it will form a Solid call'd a *Parabolick Spindle*, constituted of an infinite Series of $\odot ma$, $\odot ne$, $\odot py$, &c. by Definition 18.

Let us suppose the Line sd , parallel to AB , &c. (as at Theorem 23) then it hath already been prov'd, that the Lines fm , gn , hp , &c. are a Series of Squares whose Roots are in *Arithmetick Progression*: consequently their Squares, viz. $\square fm$, $\square gn$, $\square hp$, &c. will be a Series of Biquadrates, whose Roots will be in *Arithmetick Progression*: which being premis'd, we may proceed thus.

First, $\left\{ \begin{array}{l} 1 \mid SA - fm = ma \\ 2 \mid SA - gn = ne \\ 3 \mid SA - hp = py \text{ \&c.} \end{array} \right.$



$$\begin{array}{l|l|l} 1 \odot^2 & 4 & \square SA - 2SA \times fm + \square fm = \square ma \\ 2 \odot^2 & 5 & \square SA - 2SA \times gn + \square gn = \square ne \\ 3 \odot^2 & 6 & \square SA - 2SA \times hp + \square hp = \square py, \&c. \end{array}$$

In these *Æquations* the $\square SA$, $\square SA$, $\square SA$ being a *Series* of *Equals*, and AB the *Number* of all the *Terms*; therefore it will be $\square SA \times AB =$ the *Sum* of the *Series*, by *Lemma 1*.

2. Because fm , gn , hp , &c. are as a *Series* of *Squares* where-
in SA is the *greatest Term*, and AB the *Number* of all the *Terms*;
therefore $\frac{2SA \times SA \times AB}{3} = \frac{2 \square SA \times AB}{3}$ will be the *Sum* of all that
Series, by *Lemma 3*.

3. And the $\square fm$, $\square gn$, $\square hp$, &c. will be a *Series* of
Terms in the *Ratio* of *Biquadrates*, as above; $\square dB = \square SA$
being the *greatest Term*, and AB the *Number* of all the *Terms*;
therefore it will be $\frac{\square SA \times AB}{5} =$ the *Sum* of all that *Series*, by
Lemma 5.

Whence it follows, that $\square SA \times AB - \frac{2 \square SA \times AB}{3} +$
 $\frac{\square SA \times AB}{5} =$ the *Sum* of all the *Series* of $\square ma$, $\square ne$, $\square py$,
&c. That is, $\frac{8 \square SA \times AB}{15} =$ the *Sum* of all the *Series* of $\square ma$,

$\square ne$, $\square hp$, $\square dB$. &c. consequently, $\frac{8 \odot SA \times AB}{15} =$ the *Sum*
of all the *Series* of $\odot ma$, $\odot ne$, $\odot py$, &c which do constitute
the *Solidity* of half the *Spindle*, viz. of SAB . Therefore putting
 $D = 2SA$, and $H = 2AB$, (viz. bAB) it will be $0,41888 DDH$
 $=$ the *Solidity* of the whole *Parabolick Spindle* bSB , being $\frac{3}{5}$ of
 $0,7854 DDH$ the *Solidity* of its *circumscribing Cylinder*. Q. E. D.

From hence we may also raise a *Theorem* for finding the *Fruustum*
 $SApy$ of the last Figure. For $\odot SA$ being the *greatest Term*,
 $\odot py$ the *least Term*, and Ay the *Number* of all the *Terms* or *Cir-*
cles included between A and y ,

$$\begin{array}{l|l|l} \text{Therefore} & 1 & \left\{ \square SA - \frac{2SA \times bp}{3} + \frac{\square bp}{5} \times Ay = z \text{ the Sum} \right. \\ & & \left. \text{of all the Series } \square SA, \square ma, \square gn, \square py \right. \\ & 2 & \left\{ 3 \square SA - 2SA \times bp + \frac{3 \square bp}{5} \times Ay = 3z \right. \end{array}$$

$$2 \div Ay$$

$$\begin{array}{lcl}
 2 \div Ay & | & 3 \quad 3 \square SA - 2 SA \times bp + \frac{3 \square bp}{5} = \frac{3z}{Ay} \\
 \text{But} & , & 4 \quad \square SA - 2 SA \times bp = \square py - \square bp, \text{ by 6th Step.} \\
 3 - 4 & & 5 \quad 2 \square SA + \frac{3 \square bp}{5} = \frac{3z}{Ay} - \square py + \square bp \\
 5 + \&c. & & 6 \quad 2 \square SA + \square py - \frac{2}{3} \square bp = \frac{3z}{Ay}
 \end{array}$$

Conteq. 7 $2 \odot SA + \odot py - \frac{2}{3} \odot bp \times \frac{1}{2} Ay = z$, the *Sum* of all the *Series* of $\odot SA$, $\odot ma$, $\odot ne$, $\odot py$, which do constitute the *Solidity* of the *Frustrum* $SA py$. Therefore putting $D = 2 SA$, as before, $C = 2 py$, $x = 2 bp$, and $H = Ay$, it will be $1,5708 DD + 0,7854 CC - 0,31416 xx \times \frac{1}{3} H =$ the *Frustrum* $SA py$. And if we make $L = 2 H$. Then $1,5708 DD + 0,7854 CC - 0,31416 xx \times \frac{1}{3} L =$ Double of that *Frustrum*, being the *middle Zone*. And by turning these *Factors* into one common Divisor, as in the *Frustrum* of the *Conoid* at *Theorem 25*, Page 430, there will arise this following *Theorem*.

T H E O R E M XXVII.

$$\left\{ \begin{array}{l} 3,8196) 2 DD + CC - 0,4xx \times L (= \\ \text{the middle Zone of a Parabolick Spindle.} \end{array} \right.$$

It may be here expected that I should now proceed to shew how the *Area* of any *Hyperbola*, and the *Contents* of such *Solids* as may be form'd by the *Rotation* of that Figure about its *Axis*, &c. may be found; but because those Things cannot be exactly perform'd by any certain or settled *Theorem*, as these of the *Circle*, *Ellipsis*, and *Parabola* have been, I have therefore omitted them, and refer the Reader to Dr. *Wallis's Algebra*, Chap. 90, &c. or to the *Philosoph. Transact.* Numb. 34, wherein he may find the Method of forming *infinite Series* relating to the squaring of an *Hyperbola*, &c. which are too tedious to be fully explain'd and demonstrated in this small Tract, it being only intended as an *Introduction*, the which I shall here conclude.

F I N I S.

A N

A P P E N D I X

O F

Practical Gauging.

TH E Art of *Gauging* is that Branch of the Mathematicks called *Stereometry*, or the Measuring of Solids, because the Capacities or Contents of all Sorts of Vessels used for Liquors, &c. are computed as tho' they were really solid Bodies; which any one that hath made himself Master of the 'foregoing Parts of this Treatise may easily understand, without any farther Directions.

However, because 'tis not to be suppos'd that every one, who designs to undertake the Office or Employment of a Gauger, hath made so great a Progress in Mathematical Learning, I have therefore presented the young Gauger with this Appendix, wherein I have only inserted such Rules as are useful in Gauging, and have been already demonstrated in this Treatise. But herein, I presuppose that he hath acquir'd (or if not, 'tis very necessary he should acquire) a competent Knowledge both in Arithmetick and Geometry: That is,

I. In Arithmetick he should understand the principal Rules very well, especially Multiplication and Division, both in whole Numbers and Decimal Parts, (which may be easily learnt out of the 2d, 3d, and 5th Chapters of Part 1.) that so he may be ready at computing the Contents of any Vessel, and casting up his Gauges by the Pen only, *viz.* without the Help of those Lines of Numbers upon Sliding Rules, so much applauded, and but too much practis'd, which at best do but help to guess at the Truth; I mean such Pocket-Rules as are but nine Inches (or a Foot) long, whose Radius of the double Line of Numbers is not six Inches; and therefore the Graduations or Divisions of those Lines are so very close, that they cannot be well distinguish'd. 'Tis true, when the Rules are made two or three Feet long (I had one of six Feet) there they may be of some Use, especially in small Numbers; altho' even then the Operations may be much better (and almost as soon) done by the Pen: For, indeed, the chief Use of Sliding-Rules is only in taking of Dimensions, and for that Purpose they are very convenient.

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II. In

II. In Geometry the Gauger should understand not only how to take Dimensions (which is best learnt by Practice) but also how to divide any irregular Figure or Superficies, as Brewers Backs or Coolers, &c. into the easiest and fewest regular Figures they will admit of, that so their Area's may be truly computed with the least Trouble. And this may be learn'd (with a little Care and Diligence) out of the 1st, 2d, and 5th Chapters of Part III, which the Gauger should be well acquainted with. Also he ought to have so much Skill in Solids, as to be able, even at sight (but this must be acquired by Experience) to determine what sort of Figure any Vessel is of (*viz.* any Tun or close Cask) or what Figures it may be best reduced to, so that its Dimensions may be truly taken, and the Content thereof computed with the least Error. I say, with the least Error, because 'tis very difficult, if not impossible, to do it exactly; for there is not any Tun, or close Cask, &c. so regularly made, as by the Rules of Art 'tis requir'd to be.

III. Besides the aforementioned, the young Gauger must know, that all Dimensions useful in Gauging are to be taken in Inches, and Decimal Parts of an Inch; and if they are taken in any other Measures, as Feet, Yards, &c. those Measures must be reduced to Inches, (see Sect. 4. Page. 42.) because the Contents of all Sorts of Vessels (taken Notice of in Gauging) are computed by the Standard Gallon of its Kind, whose Content is known to be a certain Number of Cubick Inches: That is, the Beer or Ale Gallon contains 282, the Wine 231, and the Corn Gallon 268, 8 Cubick Inches. [See the five Tables, &c. in Pages 34, 35, 36, which I here suppose the Gauger to have learnt perfectly, by heart.] Consequently, if either the superficial or solid Content of any Vessel, as Back, Tun, Cask, &c. be once computed in Cubick Inches, 'twill be easy to know how many Gallons, either of Ale, Wine, or Corn, that Vessel will hold.

Note, I have here said, the Superficial Content in Cubick Inches, which may seem to be very improper, according to the Definition given of a Superficies in Page 279; but you must know, that, in the Business of Gauging, all Superficies or Area's are always understood to be one Inch deep, otherwise it could not be said (as in the Gauger's Language it is) that the Area of such a Back, or of such a Circle, &c. is so many Gallons.

These Things being very well understood, the young Gauger will be fitly prepar'd to understand the following *Problems*, which are such as have (most of them) been already propos'd in the foregoing Parts of this Treatise, and only are here apply'd to Practice; and therefore I shall, for Brevity's Sake, often refer to those Theorems and Problems.

Sect.

Sect. 1. To find the *Area* of any right-lined Superficies in Gallons.

P R O B L E M I.

To find the *Area* of any square Tun, Back, or Cooler, &c. either in Ale, Wine, or Corn Gallons.

RULE. { Multiply the given Length or Breadth (being here equal) into itself, and the Product will be the *Area* in Inches; then divide that *Area* by 282, or 231, or 268,8 and the Quotient will be the *Area* requir'd.

Example. Suppose the Side of a Square Tun, Back, or Cooler be 124,5 Inches, what will its *Area* be in Gallons?

First $124,5 \times 124,5 = 15500,25$ the *Area* in Inches.

Then 282 }
And 231 } $15500,25$ { $\begin{matrix} 54,96 \text{ } \mathcal{E}c. \\ 76,10 \text{ } \mathcal{E}c. \\ 57,66 \text{ } \mathcal{E}c. \end{matrix}$ } the *Area* in { $\begin{matrix} \text{Ale Gallons.} \\ \text{Wine Gallons} \\ \text{Corn Gallons.} \end{matrix}$

But if any one would rather work by Multiplication than by Division, he may turn or change any Divisor into a Multiplier, if he divide Unity, or 1, by that Divisor. (Vide Probl. 3, Pag. 402.)

Thus 282 }
And 231 } $1,000000$ { $\begin{matrix} 0,003546 \\ 0,004329 \\ 0,003722 \end{matrix}$ } the Multipli. for { $\begin{matrix} \text{Ale Gallons.} \\ \text{W. Gallons.} \\ \text{C. Gallons.} \end{matrix}$

Consequently $15500,25 \times 0,003546 = 54,96 \text{ } \mathcal{E}c.$ the *Area* in Ale Gallons; as before and so on for the rest.

P R O B L E M II.

To find the *Area* of any Tun, Back, or Cooler in the Form of a Right-angled Parallelogram in Ale Gallons, &c.

See the Rule for finding its *Area* in Inches, at Probl. 1. P. 339, then either divide (or multiply) that *Area* as above, and you will have the *Area* in Gallons.

Example. Suppose the Length of a Brewer's Tun, Back, or Cooler be 217,5 Inches, and its Breadth 85,6 Inches, what will its *Area* be in Ale or Beer Gallons, &c?

First $217,5 \times 85,6 = 18648$. Then 282) 18648 (66,12, &c.
Or $18648 \times 0,003546 = 66,12, \text{ } \mathcal{E}c.$ the *Area* requir'd, &c.

P R O B L E M III.

To find the Area of any Triangular Tun, Back, or Cooler, in Ale Gallons, &c.

See the Rule for finding its Area in Inches at Prob. 3, p. 340; then divide (or multiply) that Area as before, and you will have the Area requir'd.

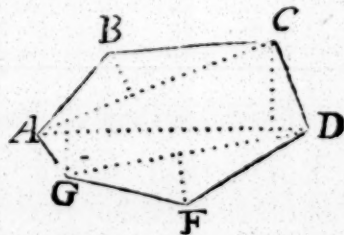
Example. If the Length of the Base of a Triangular Cooler be 86,4 Inches, and its Perpendicular Breadth be 57 Inches, what will its Area be in Ale Gallons?

First, $86,4 \times 57 = 2462,4$. Then $282) 2462,4 (8,73 \text{ \&c.}$
Or $2462,4 \times 0,003546 = 8,73 \text{ \&c.}$ the Area in Ale Gallons.

Proceeding thus, you may easily find the Area of any Tun, Back, or Cooler, whether it be in the Form of a Rhombus, Rhomboides, Trapezium, or any other Polygon, either regular or irregular, in Ale or Beer Gallons, &c. if you first divide it into Triangles, and then find the Area's of those Triangles; (as in the 2d, 4th, 5th, and 6th Problems in Chap. 5, Part III.) the Sum of those Area's being divided (or multiply'd) by its proper Divisor (or Multiplier) as above, will give the Area requir'd.

Now, the Practical Way of dividing any Polygonous Tun, Back, &c. into Triangles, is by help of a chalk'd Line, such as the Carpenters use, and may be thus perform'd.

Suppose any Brewer's Tun, Back, or Cooler in the Form of the annex'd Figure *ABCD FG*. Let one End of the chalk'd Line be fasten'd with a Nail (or otherwise) in any Corner or Angle of the Back, as at *A*; then straining it to the Angle at *C*, strike the Diagonal Line *AC*, upon the Bottom of the Back; and straining it again to the Angle *D*, strike another Diagonal Line, as *AD*, and so on for the Diagonal Line *GD*, &c. Then having mark'd out all the Diagonals, the Perpendiculars may be thus found: Fasten (as before) one End of the chalk'd Line in the Angle *B*, and then, by moving it to and fro upon the Stretch, find out the nearest Distance between the Angle at *B* and the Diagonal Line *AC*; and there strike a Line, and it will mark out the Perpendicular from *B* to the Line *AC*, and so on for the other Perpendiculars: Which being all mark'd out upon the Bottom of the Back, measure them and each Diagonal by a Line of



Inches, &c. and then the Area of that Back may be computed, as directed above.

And here, by the Way, it may be observed, that the Number of Triangles will always be less by two, and the Number of the Diagonals less by three, than the Number of the Sides of any Right-lin'd Figure that is so divided.

Having found (as above) the true Area of any Brewer's Back or Cooler (which, according to the Laws of Excise, ought always to be fix'd or immoveable) the next Thing will be to find out the true dipping or gauging Place in that Back, that so the true Quantity of Worts may be computed or (cast up) at any Depth; which may be thus done.

1. When the Bottom of the Back is covered all over (of any Depth) either with Worts or Liquor (*viz. Water*) then dip it in eight or ten several Places (more or less according to the Largeness of the Back) as remote and equally distant one from another as you well can, noting down the wet Inches and decimal Parts of every Dip.

2. Divide the Sum of all those Dips or wet Inches by the Number of Places you dipp'd in, and the Quotient will be the mean Wet of all those Dips.

3. Lastly, find out such a Place by the Side of the Back (if you can) that just wets the same with that mean Dip, and make a Notch or Mark there, for the true and constant Dipping-place of that Back. Then if any Quantity of Worts (which do cover the whole Back) be dipp'd or gaug'd at that Place, and the wet Inches so taken be multiply'd into the Area of the Back in Gallons, the Product will shew what Quantity (*viz. how many Gallons*) of Worts are in that Back at that Time, provided the Sides of the Back do stand at Right Angles with its Bottom.

Sect. 2. *To find the Area of any Circular and Elliptical Superficies in Gallons.*

1. I have demonstrated in Cap. 6, Part III, and Theorem 3, 5, 6. Part V. that the Periphery of the Circle whose Diameter is Unity, or 1, is 3,14159265 &c. (or for common Use 3,1416) and that its Area is 0,78539816 &c. (or 0,7854 *ferè*.)

2. Also, that the Peripheries of all Circles are in Proportion one to another as their Diameters are; and their Area's are in Proportion to the Squares of the Diameters. That is, as 1 : 3,1416 :: the Diameter of any Circle : to its Periphery. And 1 : 0,7854 :: the Square of the Diameter : to the Area.

Upon

Upon these two Proportions depend the Solutions of all the common or practical Questions about a Circle. [See Page 408, 409.]

P R O B L E M IV.

The Diameter of any Circle being given in Inches, to find the Periphery.

RULE. { Multiply the given Diameter with 3,1416, and the Product will be the Periphery requir'd. [See Prob. I. p. 408.]

Example. Suppose the Diameter of a Circle be 54,5 Inches, and it were requir'd to find its Periphery. Then $54,5 \times 3,1416 = 171,21$, &c. Inches is the Periphery requir'd. The Converse of this is easy, viz. by having the Periphery given, to find the Diameter. [See Prob. 3. Page 408.]

P R O B L E M V.

The Diameter of any Circle being given (in Inches) to find its Area in Gallons.

RULE. { Multiply the Square of the propos'd Diameter into 0,7854, and the Product will be the Area in Inches; [See Probl. 2, P. 408.] that Area being divided by 282, or 231, &c. the Quotient will be the Area required.

Example. Suppose the given Diameter be 54,5 Inches as above. First, $54,5 \times 54,5 = 2970,25$. And $2970,25 \times 0,7854 = 2332,83$ the Area in Inches:

Then 282 }
And 231 } 2332,83 { $\begin{matrix} 8,2724 \\ 10,0988 \\ 8,6788 \end{matrix}$ } the Area in { $\begin{matrix} \text{Ale or Beer Gallons.} \\ \text{Wine Gallons.} \\ \text{Corn Gallons.} \end{matrix}$

But these Area's in Gallons may be much easier found without knowing the Circle's Area in Inches, as above, by having the Square of the Diameter of that Circle whose Area is one Gallon; which may be thus found, by Theorem 6, Page 407.

$0,785398 : 1 :: 282 : 359,05$ the Square of the Diameter of the Circle whose Area is 282 cubick Inches, viz. one Ale Gallon.

And from this Proportion will arise the following Divisors;

viz. $0,785398 \left\{ \begin{matrix} 282,000000 (359,05) \\ 231,000000 (294,12) \\ 268,800000 (342,24) \end{matrix} \right\}$ will be a Divisor for { $\begin{matrix} A. G. \\ W. G. \\ C. G. \end{matrix}$

If

If the Square of the Diameter of any Circle be divided by any one of these constant or fixed Divisors, the Quotient will shew that Circle's Area in their respective Gallons. As for Instance, in the last Circle, whose Square of its Diameter is 2970,25.

Then 359,05 }
And 294,12 } 2970,25 { $\begin{matrix} 8,2725 \\ 10,0988 \\ 8,6788 \end{matrix}$ } the Area in $\begin{cases} A. G. \\ W. G. \\ C. G. \end{cases}$ as before.
Or 342,24 }

Now these Divisors may be turn'd into Multipliers by dividing Unity or 1, as in Page 435: Or rather by dividing the Area in Inches of that Circle whose Diameter is 1.

That is, 0,785398 by 282. Or by 231, &c.

Thus 282 }
And 231 } 0,785398 { $\begin{matrix} 0,002785 \\ 0,003399 \\ 0,002922 \end{matrix}$ } the Multiplier for $\begin{cases} Ale Gal. \\ Wine Gal. \\ Corn Gal. \end{cases}$
Or 268,8 }

These Multipliers are the respective Area's of a Circle whose Diameter is 1; and therefore, if the Square of the Diameter of any Circle be multiply'd with any of these Numbers, the Product will be that Circle's Area in Gallons of the same Name:

Viz. $2970,25 \times 0,002785 = 8,2725$ the Area in *A. G.* as above.
And $2970,25 \times 0,003399 = 10,0988$ the Area in *W. Gal.* &c.

Thus you see, that if the Diameter of any Circle be given in Inches, there are three several Ways of finding its Area in Gallons, and all equally true; but that which is perform'd by the constant Divisors is most generally practis'd.

PROBLEM VI.

The Transverse (or longest Diameter) and the Conjugate (or shortest Diameter) of any Elliptical Superficies being given, to find its Area in Gallons.

RULE. $\begin{cases} \text{Multiply the two Diameters (viz. the Length and} \\ \text{Breadth) together, and divide their Product by 359,05} \\ \text{for Ale Gallons, or 294, 12 for Wine Gallons, \&c. the} \\ \text{Quotient will be the Area requir'd. [See Theorem 7,} \\ \text{Page 412.} \end{cases}$

Example. Suppose the longest Diameter to be 73,5 Inches and the shortest Diameter to be 51,6 Inches; what will the Area be in Ale Gallons?

First $73,5 \times 51,6 = 3792,6$. Then 359,05) 3792,6 (10,56 the Area in Ale Gallons. Or 294,12) 3792,6 (12,89 the Area in Wine Gallons, &c.

Note,

Note, The two last Problems are of a great Use in Gauging of Worts amongst Country Victuallers, who generally brew but short Lengths of Ale (perhaps between 20 and 60 Gallons at a Brewing) and cool their Worts in several small open Vessels or Tubs, whose Bases or Bottoms are either a Circle, or an Ellipsis, having their Sides but low, and are most commonly wider at the Top than at the Bottom.

Now a practical Way of computing the Quantity of Worts, that are at any Time in one of those open Tubs, is briefly thus: When the Tub is dry, find the true Area of its Bottom according to its Figure (as above) and either mark that Area on the Outside of the Tub (which was the Way I generally us'd to order, because the Victuallers did often lend their cooling Tubs one to another) or else number the Tub, and enter its Area (and its Number) into the Stock-book; then, when any of those Tubs hath Worts in it, take the Diameter of the Surface or Top of the Worts, and find that Area, adding it and the bottom Area together. If either the half Sum of those two Area's be multiply'd with the Depth of the Worts (taken as near the Middle of the Tub as you well can) or, if the Sum of those two Area's be multiply'd with half the Depth (so taken) the Product will shew the Quantity of those Worts very near the Truth.

P R O B L E M VII.

The Diameter of any Circle, and the versed Sine, viz. (the Height of any Segment, being given, to find the Area of that Segment in Gallons.

In the 410th and 412th Pages you have two Ways (and their Examples) of finding the Area of any Segment of a Circle in Inches; then if that Area in Inches be divided by 282, or 231 &c.) the Quotient will be its Area in Gallons. But because the Area of any such Segment may be readily found in Gallons (without finding its Area in Inches) by help of a Table of Segments, whose Construction is laid down in the Problem, Page 411, &c. I have here inserted a Compendium of such a Table, which will serve very well for common Practice, not only to find the Area of any Segment of a Circle in Gallons, but also to find the Number of Gallons that are either drawn out, or remaining in any Cylandrick Vessel lying along; or of any close Cask (being first reduced to a Cylinder) its Axis lying parallel to the Horizon, usually call'd the *Ullage* of a Cask; as shall be shew'd farther on.

A Table

A Table of the Segments of a Circle whose Area is Unity or 1, the Diameter being divided by parallel Chord-Lines into 100 equal Parts.

V.S.	Segment.	V.S.	Segment.	V.S.	Segment.	V.S.	Segment.
1	0,0017	26	0,2066	51	0,5127	76	0,8155
2	0,0048	27	0,2178	52	0,5255	77	0,8262
3	0,0087	28	0,2292	53	0,5382	78	0,8369
4	0,0134	29	0,2407	54	0,5509	79	0,8474
5	0,0187	30	0,2523	55	0,5635	80	0,8576
6	0,0245	31	0,2640	56	0,5762	81	0,8677
7	0,0308	32	0,2759	57	0,5888	82	0,8776
8	0,0375	33	0,2878	58	0,6014	83	0,8873
9	0,0446	34	0,2998	59	0,6140	84	0,8968
10	0,0520	35	0,3119	60	0,6265	85	0,9059
11	0,0598	36	0,3241	61	0,6389	86	0,9149
12	0,0680	37	0,3364	62	0,6514	87	0,9236
13	0,0764	38	0,3486	63	0,6636	88	0,9320
14	0,0851	39	0,3611	64	0,6759	89	0,9402
15	0,0941	40	0,3735	65	0,6881	90	0,9480
16	0,1032	41	0,3860	66	0,7002	91	0,9554
17	0,1127	42	0,3986	67	0,7122	92	0,9625
18	0,1224	43	0,4112	68	0,7241	93	0,9692
19	0,1323	44	0,4238	69	0,7360	94	0,9755
20	0,1424	45	0,4365	70	0,7477	95	0,9813
21	0,1526	46	0,4491	71	0,7593	96	0,9866
22	0,1631	47	0,4618	72	0,7708	97	0,9913
23	0,1738	48	0,4745	73	0,7822	98	0,9952
24	0,1845	49	0,4873	74	0,7934	99	0,9983
25	0,1955	50	0,5000	75	0,8045	100	1,0000

The Use of this Table of Segments depends upon the following Proportion,

viz. { As the Diameter of any propos'd Circle : is to 100 (the Diameter of the tabular Circle) :: so is the Height of any Segment of the propos'd Circle : to a versed Sine in the Table.

Then, if the tabular Segment, which stands against that versed Sine, be multiply'd into the Circle's Area (either in Inches or Gallons) the Product will be the Area of the Segment requir'd [of the same Name] *viz.* If the Circle's Area be Inches, the Segment will be Inches; if Gallons, the Segment will be Gallons.

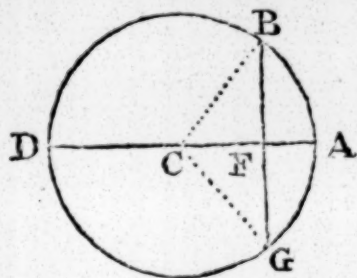
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Example.

Example. Let the Diameter of the given Circle be $DA = 62,5$ Inches, and the Height of the Segment sought be $FA = 20$ Inches; What will its Area be in Ale Gallons?

First, the Area of the whole Circle will be 10,8793 Ale Gallons (by Problem 5.) and the Proportion will stand thus, $62,5 : 100 :: 20 : 32$ the versed Sine of the Table whose Segment is 0,2759. Then, $10,8720 \times 0,2759 = 3,$

0016 Ale Gallons, being the Area of the Segment $BAGF$, as was requir'd. The like may be done for Wine Gallons, Corn Gallons, or Inches.



And, upon Occasion, the like Segments of any Ellipsis may be easily found. See the *Proportions* in the Corollaries to the 7th and 8th Theorems, Page 412, &c. to which I here, for Brevity's Sake, refer the Reader.

Se^{ct}. 3. To compute the Contents of such Vessels (*viz.* Tuns, &c.) as are in the Form of the following Solids.

Note, Before the young Gauger proceeds to these Computations, he should be well acquainted with such Solids as are defin'd in P. 402 and 403, and then he may easily understand what Sort of Figures are meant in the following Problems, without the Repetition of many Words.

P R O B L E M VIII.

To find the Content of any Prism whose Sides are Parallelograms what Form soever its Base is of.

That is, to compute the Content (in Gallons) of any Tun, &c. whose Sides are Parallelograms which stand upright, or at Right Angles with its Bottom.

First, find its solid Content in Inches, by Theorem 9, Page 414; then divide that Content by 282, or 231, or by 268,8; the Quotient will shew the Content in their respective Gallons, *viz.* in Ale, Wine, or Corn Gallons.

Or else multiply the Content in Inches with 0,003546, or 0,004329, &c. [See the Multipliers, Page 435] those Products will be the Content in their respective Gallons.

Or otherwise thus:

Find the true Area of the Tun's Base or Bottom, as directed in Se^{ct}. 1, P. 435; that Area being multiply'd with the Tun's Height (*viz.* Depth within) will produce the Content in Gallons, as before.

I take

I take the Work of this Problem to be so very easy, that it needs no Example.

P R O B L E M IX.

To find the Content of any Pyramid (in Gallons) whose Base is bounded with Right Lines.

Every Pyramid is one Third-part of its circumscribing Prism, by Theorem 10, Page 415. Therefore, if the Area of the Base of any Pyramid, in Gallons, be multiply'd in one Third of its perpendicular Height; or if one Third of that Area be multiply'd with the whole Height, either of those Products will be the Content of the Pyramid in Gallons, &c. But the Content of any square Pyramid may be easily found in Gallons by this Rule:

RULE. { Square the Side of its Base, and multiply that Square with the perpendicular Height; then divide that Product by $846 = 282 \times 3$ for Ale Gallons, or by $693 = 231 \times 3$ for Wine Gallons, or by $806,4 = 268,8 \times 3$ for Corn Gallons, the Quotient will be the Content requir'd.

Or, if you multiply the said Product with 0,001182 for *AG.* or with 0,001443 for *WG.* or, lastly, with 0,001241 for *C. G.* the Result will be the Content requir'd, as before.

P R O B L E M X.

To find the Content (in Gallons) of the Frustrum of any square Pyramid, cut off by a Plain parallel to its Base.

First, either by Theorem 15, Page 419, or Theorem 16, P. 420, find the propos'd Frustrum's Solidity in Cubick Inches; then divide that Content in Cubick Inches by 282 or 231, &c. and the Quotient will be the Content of the Frustrum in their respective Gallons.

But, from the foresaid Theorem 15, there may be easily deduced the following general Rule for finding the Content of the like Frustrum of any Pyramid, what Form soever its Bases are of (supposing them to be parallel) whether they are alike or unlike.

RULE. { First, find the Area of each Base, (*viz.* the top and bottom Area's of the propos'd Frustrum;) then find a Geometrical Mean between those two Area's (by Lemma 1, Page 83;) the Sum of those two Area's and their Mean, being multiply'd into one Third of the Frustrum's Height, will produce the Content required.

Example. Suppose a Tun in the Form of the lower Fruustum of a Pyramid, whose Bases are Equilateral Triangles: Let the Side of the Top be 42 Inches, the Side of the Bottom be 63,4 Inches, and its Height [*viz.* Depth] be 33 Inches; What will the Content of that Tun be in Ale Gallons?

First, find the Area of that Base in Inches, by Probl. 7, P. 343; then find what those Area's are in Ale Gallons, by Probl. 3, P. 436. Multiply those two Area's together and the square Root of their Product will be the mean Area, &c. as in this Example:

Example. The $\left\{ \begin{array}{l} \text{Top} \\ \text{Bottom} \\ \text{Mean} \end{array} \right\}$ Area is $\left\{ \begin{array}{l} 2,71 \\ 6,12 \\ 4,07 \end{array} \right\}$ Ale Gallons.
 Their Sum 12,90

Then $12,9 \times \frac{23}{3} = 141,9$. Or $\frac{12,9}{3} \times 33 = 141,9$ the Content required.

P R O B L E M X I.

To find the Content of any right Cylinder in Gallons.

That is, to compute the Content of any round Tun, &c. whose Diameters at Top and Bottom are equal, and at Right Angles with its Sides.

The Content of such a Tun may be found by Theorem 11, Page 415; or otherwise by the following Rule.

RULE. $\left\{ \begin{array}{l} \text{Multiply the Square of the Diameter into the Height,} \\ \text{and divide the Product by 359,05 (or multiply with} \\ \text{0,002785) \&c. as in Page 439, that Quotient (or Pro-} \\ \text{duct) will be the Content required.} \end{array} \right.$

Exam. Suppose the Diameter be 42,5, and the Height 31,5 Inches. First $42,5 \times 42,5 = 1806,25$. And $1806,25 \times 31,5 = 56896,875$. Then $359,05 \mid 56896,875$ (158,46 the Content in Ale Gal. &c.

P R O B L E M X II.

To find the Content of any Cone or round Pyramid in Gallons.

Because every Cone is one Third of its circumscribing Cylinder, [See Theorem 13, Page 416] therefore its Content may be truly found by the following Rule.

RULE. $\left\{ \begin{array}{l} \text{Multiply the Square of the Diameter of its Base into} \\ \text{the perpendicular Height, then divide their Product} \\ \text{by } 1077,15 = 359,05 \times 3 \text{ for Ale Gallons, or by} \\ \text{882,36} = 294,12 \times 3 \text{ for Wine Gallons, \&c. and the} \\ \text{Quotient will be the Content required.} \end{array} \right.$

Of

Or if the said Product be multiply'd with $0,000928 = \frac{0,002785}{3}$,

or with $0,001133 = \frac{0,0034}{3}$, those Products will be the Content in their respective Gallons.

Example. Suppose the Diameter of the Base be 42,5, and the perpendicular Height be 31,5 Inches, what will the Content be in Ale Gallons? (as before.)

First $42,5 \times 42,5 = 1806,25$. And $1806,25 \times 31,5 = 56896,875$
Then $1077,15 \mid 56896,875$ (52,82. Or $56896,25 \times 0,000928 = 52,82$ the Content in Ale Gallons. And so on for Wine or Corn Gallons.

PROBLEM XIII.

To find the Content of the lower Frustrum of any Cone in Gallons.

That is, to compute the Content of any round Tun, &c. whose Diameters at Top and Bottom are parallel, but unequal.

The Content of such a Tun may be found by the Rule at Problem 10; but from Theorem 16, Page 420, 'twill be easy to deduce this following Rule.

RULE. { To the triple Product of the Top and bottom Diameters, add the Square of their Difference; multiply that Sum into the Height (or Depth): then divide the last Product by 1077,15 for Ale Gallons, or by 882,36 for Wine Gallons; the Quotient will be the Content requir'd.

Example. Suppose the Diameter at the Top to be 52,4 Inches, the Diameter at the Bottom 44,6, and the Height 30 Inches.

First, $52,4 \times 44,6 = 2337,04$; and $2337,04 \times 3 = 7011,12$ } Add
Also, $52,4 - 44,6 = 7,8$; and $7,8 \times 7,8 = 60,84$ }
The Height 30 $\times$ $7071,96 = 212158,8$.

Then $1077,15 \mid 212158,8$ (196,96 } the Content in Ale Gallons.
Or $212158,8 \times 0,000928 = 196,96$ }

And so on for either Wine or Corn Gallons, as Occasion requires. But if the Tun (or Vessel) be not truly circular, that is, if either its Top or Bottom (or both of 'em) be Elliptical, whether they are alike or unlike, it matters not, the Content of such a Tun may be truly found by the general Rule at Problem 10.

PROBLEM XIV.

The Axis or Diameter of any Sphere or Globe being given in Inches to find its Content in Gallons.

Every Sphere is two Thirds of its circumscribing Cylinder, by Theor. 18, Page 423; from whence and Theor. 20, Page 426, 'tis proved,

proved, that if the Cube of the Axis of any Sphere (taken in Inches) be multiply'd into 0,5236, the Product will be the Content of that Sphere in Inches. Consequently, if that Content be divided by 282, or by 231, &c. the Quotient will be the Content in Gallons.

But those two Works of multiplying with 0,5236, and then dividing by 282, or by 231, &c. may be contracted into one.

Thus $282 \left\{ \begin{array}{l} 0,5236 \\ 0,001856 \end{array} \right\}$ will be a *Multiplicator* for $\left\{ \begin{array}{l} A. G. \\ W. G. \end{array} \right\}$.

And $231 \left\{ \begin{array}{l} 0,5236 \\ 0,002266 \end{array} \right\}$ will be a *Multiplicator* for $\left\{ \begin{array}{l} A. G. \\ W. G. \end{array} \right\}$.

Or $0,5236 \left\{ \begin{array}{l} 282 \\ 231 \end{array} \right\} \left\{ \begin{array}{l} 538,57 \\ 441,17 \end{array} \right\}$ will be a *Divisor* for $\left\{ \begin{array}{l} Ale Gallons. \\ Wine Gallons. \end{array} \right\}$.

From hence arises this following Rule.

RULE. If the Cube of the Axis of any Sphere be divided by 538,57; or multiply'd with 0,001856: or divided by 441,17; or else multiply'd with 0,002266; the Quotient, or Product, will be the Sphere's Content in their respective Gallons.

Example. Suppose the Axis or Diameter of a Sphere or Globe be 2 Inches, how many Ale Gallons may it hold?

Here $22 \times 22 \times 22 = 10648$; and $538,57 \mid 10648$ (19,76 *A. G.*
Or $10648 \times 0,001856 = 19,76$ Ale Gal. the Content required.
And so for either Wine or Corn Gallons, as Occasion requires.

P R O B L E M XV.

To find the Content of a Segment of a Sphere in Gallons.

In the Scholium, P. 424, there are two Theorems for resolving this Problem according to the Data.

1. If the Diameter of the Segment's Base and its Height are given, the Content may be found by the first of those Theorems, which gives this Rule:

RULE 1. $\left\{ \begin{array}{l} \text{To the Triple Square of half the Diameter add the} \\ \text{Square of the Height; then multiply that Sum into} \\ \text{the Height, and divide the Product by 538,57 for} \\ \text{A G, or by 441,17 for W G, \&c. as above.} \end{array} \right\}$

2. But if the Axis of the Sphere and the Height of the Segment are given, the Content may be found by the Second of those Theorems.

RULE 2. $\left\{ \begin{array}{l} \text{From the triple Product of the Axis into the Height,} \\ \text{subtract twice the Square of the Height; then mul-} \\ \text{tiply the Remainder into the Height, and divide that} \\ \text{Product by 538,57, \&c. as in the last Problem.} \end{array} \right\}$

Either of these Rules will produce the Content of the Segment in Gallons.

Example. Suppose the Diameter of the Segment's Base be 28 Inches, and its Height be 8 Inches, what may it contain in Ale Gallons?

First 2) 28 (14. Then (by Rule 1.) $14 \times 14 \times 3 = 588$. And $6 \times 6 = 36$. Next $588 + 36 = 624$. Again $624 \times 6 = 3744$. Lastly, 538,57) 3744 (6,95 the Content required.

Note, This Problem may be of Use in Gauging the Crowns of Brewers Coppers, &c.

Señt. 4. The Practical Method of Gauging any fix'd Tun or Copper, and making a Table to shew what it will hold at every Inch deep, usually call'd Inching of a Tun, &c.

First, you must know, that most (if not all) Brewers Tuns are so fix'd as to lean a little for Conveniency of cleansing their Drink, which is usually call'd the Drip or Fall of the Tun. Now this Drip or Fall of any Tun is the Hoof of such a Solid as that Tun is suppos'd to represent, and under that Consideration it may be found, as in Theor. 16, P. 420: But the practical (and indeed the best) Way is, to measure into the Tun (when 'tis dry) so much Liquor as will just cover its Bottom; for by that means you do not only find the true Fall, but also a true horizontal or level Plain over the Bottom of the Tun; from which if the Depth of the Tun (*viz.* the nearest Distance from the Top of the Tun to the Surface of the Liquor) be set off upon every one of its Sides, you will then have a true parallel Plain at the Top of the Tun to that of the Liquor. Then, if the Sides of the Tun are streight from the Top to the Bottom, take as many Dimensions in the aforesaid two Plains as are needful to find the true Area of each; and by those two Area's and the foresaid Depth find so much of the Tun's Content (by the general Rule at Problem X.) as is betwixt those two Plains.

Next, to inch that Tun, divide the Difference between the Top and Bottom Area's by the aforesaid Depth, and the Quotient will be an Addend or fixed Number; which being added to the lesser Area, the Sum will be the Area of the next Inch; and, being added to that Area, their Sum will be the Area of the third Inch; and so on from Inch to Inch, until the Area of every single Inch be found; the Sum of those Area's (if the Work be true) will amount (or be equal) to the Content found, as above. And if
the

the Tun's Drip or Fall be added to the Sum of all those Area's, that Sum will be the whole or full Content of that Tun.

Now, from hence it must needs be easy to conceive, that if 1, 2, 3, or any Number of those Area's accounted from the Bottom, be added to the Fall, that Sum will shew the Quantity of Liquor or Drink that is in the Tun, to such a Number of wet Inches from the Bottom as there were Area's added together. Or, if the Sum of any Number of those Area's (being accounted from the Top) be subtracted from the Tun's whole Content, the Remainder will shew what Quantity of Liquor or Drink is in the Tun, when there is such a Number of dry Inches from the Top as there were Area's subtracted.

This being well consider'd, it will be easy to make a Table either to every wet or dry Inch of any regular Tun (*viz.* whose Sides are streight from Top to Bottom) what Form soever its Bases are of, and whether it stand upon the greater or lesser Base.

But if the Sides of the Tun are irregular (*viz.* not streight from its Top to the Bottom) then the best and easiest Way will be to divide or part the Tun into several Frustrums, each of ten Inches deep; and finding the Content of every single Frustrum, by taking the Diameters in the Middle of every one of those ten Inches (that is, the first Diameter at 5 Inches from the Top; the second Diameter at 15 Inches from the Top, &c.) and multiplying their respective Area's with 10, (which is done by only removing the separating Comma's one Place forward to the right Hand) if the Sum of all those Frustrums be added to the Fall, (as before); that Sum will be the whole Content of the Tun.

Note, If you take the Height of the 'foresaid ten Inch Frustrums in the Side of the Tun, you must allow for the Difference between the slant Height and the Perpendicular Height in every Frustrum.

Lastly, If from the whole Content of the Tun you subtract the mean Area of the first Frustrum ten Times, and from the Remainder subtract the mean Area of the second Frustrum ten Times, and from the last Remainder subtract the mean Area of the third Frustrum, &c. until there remain nothing but the Fall or Hoof of the Tun, you will then by that Means have a Table that will shew what Quantity of Drink is in the Tun to any Number of dry Inches.

And this is also the Method of Gauging and Inching Brewers Coppers, *viz.* by first measuring into the Copper so much Liquor as will just cover its Crown, and then dividing its perpendicular Height into Frustrums, and its Sides into four equal Parts; that so cross Diameters may be taken in the Middle of each Frustrum:

But

but if the Copper be much wider at the Top than at the Bottom, and its Sides spheroidal or arching, as generally all large Coppers are; then, instead of taking those mean Diameters in the Middle of every ten Inches, as above, you must take them in the Middle of every six Inches, and proceed on as before.

Now the Quantity of Liquor, that would cover the Crown of the Copper, may be found without measuring it, as above. In order to that, I do suppose the Crown to be the Segment of a Sphere, and the lower Part of the Copper wherein the Crown ariseth, to be the Frustrum of a parabolick Conoid; then, if the Diameter at the Top of the Crown, and its perpendicular Height are given, the Quantity of Liquor may be found by this following Rule :

RULE. { From the Area of the Plain at the Top of the Crown subtract $\frac{1}{3}$ of the Area of the Crown's Height; the Remainder, being multiply'd into half the Height of the Crown, will produce the Quantity or Number of Gallons that will cover the Crown.

This Rule is deduced from *Scholium, Page 424, and Theorem 15. Page 430.*

Seet. 5. To compute the Content of any close Cask in Gallons, viz. of any Butt, Pipe, Hoghead, Barrel, &c.

In order to perform this difficult Part of Gauging, the three following Dimensions of the proposed Cask must be truly taken in Inches, and Decimal Parts of an Inch.

Viz. { The Bulge or Bung Diameter within the Cask.
Either of the Head Diameters supposing them both equal.
And the Length of the Cask within.

Note, In taking of these Dimensions, it must be carefully observed,

1. That the Bung-hole be in the Middle of the Cask; also that the Bung-staff and the Staff over-against the Bung-hole are both regular or even within.

2. That the Heads of the Cask are equal and truly circular; if so, the Distance between the Inside of the Chine to the Outside of its opposite Staff will be the Head Diameter within the Cask, very near.

3. With a sliding pair of Calipers (made on purpose for that Use) take the shortest Distance at Length between the Outfides of the two Heads; (supposing them even) from that Length subtract $1 \frac{1}{2}$ Inch (more, or less, according to the Largeness of the

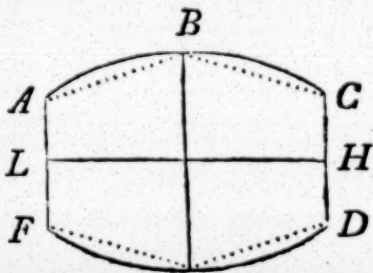
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Cask)

Cask) for the Thickness of the two Heads, the Remainder will be the Length of the Cask within.

Now by these Dimensions, one would suppose the Content of the Cask were perfectly limited; but it will be easy to perceive, by the following Figure, that the Diameters (abovesaid) and the Length of one Cask may be equal to those of another, and yet one of those Casks may contain or hold several Gallons more than the other.

As for Instance, suppose the annex'd Figure *ABCDGF*, to represent a Cask; then it is plain, that, if the outward and curved Lines *ABC* and *FGD* are the Bounds or Staves of the Cask, it must needs hold more than if the inner streight or prick'd Lines were its Bounds or Staves; and yet the Bung Diameter *BG*, Head Diameter *CD* and *AF*, and the Length *LH*, are the same in both those Casks.



Whence it plainly appears, that no one certain or general Rule can be prescrib'd to find the true Content of all Sorts of Casks, and therefore Gaugers do usually suppose every Cask to be in Form of some one of these following Solids.

- Viz.* {
- I. The middle Zone or Frustum of a Spheroid.
 - II. The middle Zone or Frustum of a Parabolick Spindle.
 - III. The lower Frustums of two equal Parabolick Conoids.
 - IV. The lower Frustums of two equal Cones.

Now the Way of Gueffing at the Cask's Form, and computing its Content, according to its suppos'd Form, I shall here shew in their Order.

I. If the Staves of the Cask are very much curved or arching (as the outward Lines of the last Figure) then the Cask is suppos'd to be in the Form of the middle Zone or Frustum of a Spheroid, whose Content may be computed, by *Theorem 22. Page 427*, which gives these two *Rules*.

RULE I. { To twice the Square of the Bung Diameter add the Square of the Head Diameter; multiply that Sum into the Length, and divide the Product by 1077,15.
Viz. 3, 8197×282 for Ale Gallons; and by 882,36.
Viz. 3, 8197×231 for Wine Gallons. Or thus,

RULE

RULE 2. { To twice the Area of the Bung Circle, add the Area of the Head Circle; multiply their Sum into one Third of the Length, and the Product will be the Content in their respective Gallons.

Example 1. Suppose a Cask in the Form of the middle Zone of a Spheroid, whose Bung Diameter is 31,5, Head Diameter 24,5, and its Length 42 Inches.

First $31,5 \times 31,5 \times 2 = 1984,5$. And $24,5 \times 24,5 = 600,25$
 Again $1984,5 + 600,25 = 2584,75$. And $2584,75 \times 42 = 108559,5$
 Then $1077,15 \mid 108559,5$ (100,78 the Content in Ale Gallons.
 And $882,35 \mid 108559,5$ (123,03 the Content in Wine Gallons.

Or thus, by the Second Rule.

Bung Diameter 31,5 twice its Circle's Area is 5,5270
 Head Diameter 24,5 its Circle's Area is 1,6718
 The Length 42 divided by 3 is 14. $7,1988 =$ their Sum.
 Then $7,1988 \times 14 = 100,78$, the Content in A. Gallons as before.
 And so the Content in Wine Gallons may be found.

II. If the Staves of the Cask are not quite so much curved or arching, as was suppos'd before, the Cask is then taken for the middle Frustrum of a parabolick Spindle, and its Content is computed, as by *Theorem 27. Page 432.* Which gives this *Rule.*

RULE. { To twice the Square of the Bung Diameter add the Square of the Head Diameter; from their Difference subtract four Tenths of the Square of the Difference of the Diameters; multiply the Remainder into the Length, and divide the Product by 1077,15, &c. as above.

Example 2. Suppose the Dimensions the same as before. Then $31,5 \times 31,5 \times 2 + 24,5 \times 24,5 = 2584,75$. And $31,5 - 24,5 = 7$. Again $7 \times 7 \times 0,4 = 19,6$. And $2584,75 - 19,6 \times 42 = 107736,3$. Then $1077,15 \mid 107736,3$ (100,01 the Cont. in A. G. &c. for W. G.

III. When the Staves of the Cask are but very little curved or arching, then it's suppos'd to be in the Form of the Frustrums of two equal parabolick Conoids, abutting or joining together upon one common Base at the Bulge, and the Content may be found by *Theorem 25. Page 430.* which gives these *Rules.*

- RULE 1. $\left\{ \begin{array}{l} \text{To the Square of the Bung Diameter add the Square} \\ \text{of the Head Diameter; multiply their Sum into the} \\ \text{Length, and divide the Product by 718,08 (viz.} \\ \text{2,5464} \times \text{282) for Ale Gallons: or by 588,22 (viz.} \\ \text{2,5464} \times \text{231) for Wine Gallons. Or thus,} \end{array} \right.$
- RULE 2. $\left\{ \begin{array}{l} \text{To the Area of the Bung Circle add the Area of the} \\ \text{Head Circle; multiply the Sum into half the Length,} \\ \text{and the Product will be the Content required.} \end{array} \right.$

Example. 3. With the same Dimensions as before. Then

$315 \times 31,5 + 24,5 \times 24,5 = 1592,5$. And $1592,5 \times 42 = 66885$
 And 718,08) 66885 (93,01 the Content in Ale Gallons.
 Or 588,22) 66885 (113,7 the Content in Wine Gallons.

IV. If the Staves of the Cask are streight from the Bulge to the Head, as the inner prick'd Lines in the last Figure (if such a Cask can be made) it is then taken for the lower Frustrums of two equal Cones, abutting or joining together upon one common Base at the Bulge. And its Content may be computed as at Problem 13. Page 445. or by Theorem 15. Page 419. Thus,

- RULE. $\left\{ \begin{array}{l} \text{To the Sum of the Squares of the Head and Bung Dia-} \\ \text{meters add their Product; then multiply that Sum into} \\ \text{the Length, and divide the last Product by 1077,15.} \\ \text{Or by 882,36. The Quotient will be the Content, \&c.} \end{array} \right.$

Example 4. With the same Dimensions as before.

First $31,5 \times 31,5 + 24,5 \times 24,5 + 31,5 \times 24,5 = 2364,25$
 And $2364,25 \times 42 = 99298,5$. Then 1077,15) 99298,5 (92,18
 the Content in Ale Gallons, and so on for Wine Gallons.

Thus you have the Methods of computing the true Contents of the four Solids, in whose Form all Casks

are suppos'd to be. And by the Exam-
 ples it appears, that four such Casks as
 have their Dimensions all equal, and the
 same with those above-mention'd, their
 Contents will be as in the Margin.

Ale Gallons.	Differ.
I. 100,78	
II. 100,01	0,77
III. 93,01	7,00
IV. 92,18	0,83

From the Disproportion or Inequality of these Differences it will be easy to conceive, that there may be several Casks whose Contents cannot be truly found, according to the aforesaid suppos'd Forms; and therefore, in order to rectify the said Inequalities, some Authors (that have written upon this Subject) have laid down Theorems of their own Invention; and yet call'd them
 by

by these Names) others have propos'd Tables for the same Purpose. But since it is so, that we can only guess at the Truth, the plainest and easiest Way is to be preferr'd in Practice; and that is, by finding such a mean Diameter as will reduce the propos'd Cask to a Cylinder.

Thus, $\left\{ \begin{array}{l} \text{Multiply the Difference between the Head and Bung} \\ \text{Diameters, with } 0,7. \text{ or with } 0,65. \text{ or with } 0,6. \text{ or with} \\ 0,55. \text{ according as the Staves of the Cask are more or} \\ \text{less arching; add the Product to the Head Diameter, and} \\ \text{the Sum will be the mean Diameter required. Then} \\ \text{find the Content, as at } \textit{Prob. 11. Page 444.} \end{array} \right.$

Example. With the same Dimensions as before. Then the Bung Diameter less the Head Diam. is $31,5 - 24,5 = 7$. And

	<i>MD.</i>	<i>AG.</i>	<i>Cont.</i>	<i>Dif.</i>
	$7 \times 0,7 = 29,40$	its <i>Area</i>	$2,4073 \times 42 = 101,10$	
$24,5 +$	$7 \times 0,65 = 29,05$	—	$2,3504 \times 42 = 98,71$	2,39
	$7 \times 0,6 = 28,70$	—	$2,2941 \times 42 = 96,35$	2,36
	$7 \times 0,55 = 28,35$	—	$2,2385 \times 42 = 94,02$	2,32

From these it may be observ'd, that the Difference between each Cask's Content is regular, and very near equal; which plainly shews, that there is not so much Room left for Error this Way of computing their Contents, as was by the aforesaid Forms.

Now the first of these four (*viz.* with $0,7$) is very commonly used among *Gaugers* for all Sorts of Casks; but I did never gauge any Cask that would contain quite so much as that Rule did make it; and the Reason doth appear very plain from *Theorem 22. Page 427.* being compar'd with *Theorem 19. Page 426.* and the last Figure; *viz.* that no (Cask being regularly made) can hold more than the middle Fusfrum of a Spheroid. But I always found by Experience, that if the second and third of these Rules (*viz.* with $0,65$ and $0,6$) were duly apply'd, they would answer very near the Truth amongst the common Sort of Casks; and the fourth Rule (*viz.* with $0,55$) will come pretty near the Truth in computing the Contents of Casks, whose Staves are almost streight betwixt the Head and Bung, *viz.* such as Wine Pipes, &c.

Seet. 6. To find what Quantity of Liquor is either drawn forth, or remaining in any spheroidical Cask, usually call'd the Ullage of a Cask; hath two Cases.

Case 1. To find what Quantity of Liquor is in the Cask, when its Axis is perpendicular to the Horizon, *viz.* when it stands upright upon one of its Heads.

In

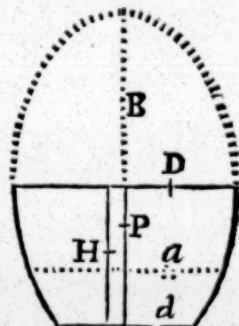
In order to perform this the easiest Way, it will be convenient to know how to calculate the Area of any Circle betwixt the Bung and Head, whose Distance from the Bung or middle of the Cask is given. Now that may be done by this Proportion.

Viz. { As the Square of half the Length of the Cask : is to the Difference between the Bung and Head Area's : : so is the Square of any Circle's Distance from the Bung : to the Difference between the Bung Area, and the Area of the Circle, viz. the Area of the Liquor's Surface.

Demonstration.

Let { H = Half the Length of the Cask
 D = Half the Bung Diameter.
 d = Half the Head Diameter.

And { P = the Distance of any Circle from the Bung
 a = Half the Diameter of that Circle.



Then according to the common Property of the Ellipsis, Page 368, it will be,

$BB : DD :: BB - HH : dd$. And $BB : DD :: BB - PP : aa$.

Ergo { $\frac{DDHH}{DD-dd} = BB$. And { $\frac{DDPP}{DD-aa} = BB$.

Consequently, { $\frac{DDHH}{DD-dd} = \frac{DD-aa}{DDPP}$.

This Equation, being brought out of the Fractions, will become $DDHH - aaHH = DDPP - ddPP$, which gives this Analogy $HH : DD - dd :: PP : DD - aa$. Then $DD - aa$, being subtracted from DD , will leave aa . But Circles Area's are in Proportion to the Squares of their Diameters, by Theorem 6, Page 407. Therefore, &c. Q. E. D. Then, from the Bung Area subtract one third Part of the aforesaid Difference, viz. between the Bung Area and the Area of the Liquor's Surface; multiply the Remainder with the Liquor's Distance from the Bung, and the Product will shew what Quantity of Liquor is either above or under half the Content of the Cask.

Example. Let us suppose a Cask of the same Dimensions with that in the first Example, Page 451. and let it be required to find what Quantity of Liquor is in it (of Ale Measure) when there is but 9 Inches wet. Here half the Length of the Cask is 21

Inches, whose Square is 441, and the Liquor's Distance from the Bung is $21 - 9 = 12$. Its Square is 144. The Difference between the Bung and Head Area's is 1,0917 ($= 2,7635 - 1,6718$.) Then $441 : 1,0917 :: 144 : 0,3564$.

And $2,7635 - 0,3564 = 2,4071$ the Area of the Liquor's Surface.

Again 3) $0,3564 (0,1188$. And $2,7635 - 0,1188 = 2,6447$ Then $2,6447 \times 12 = 31,7364$, what the Cask wants of being half full. Consequently $50,39 - 31,73 = 18,66$ will be the Quantity of Liquor in the Cask at 9 Inches wet in Ale Gallons.

And if the Cask had wanted but 9 Inches of being full; then $50,39 + 31,73 = 82,12$ would have been the Quantity of Liquor in the Cask.

Note, because the two first Terms (*viz.* 441 and 1,0917) in the Proportion are fix'd, *viz.* continue the same for any Distance, 'twill be very easy to calculate the Area's of all the Circles betwixt the Bung and Head to every Inch, and by that Means to make a Table that will shew what Quantity of Liquor is either drawn out or remaining in the Cask at any Depth.

Case 2. To find what Quantity of Liquor is in any Cask, when its Axis is parallel to the Horizon, *viz.* when it lies along.

There are Variety of Tables to be found in Books of *Gauging* for this Purpose; but I always observed, that the following Method of computing the Ullage, by a Table of the Segments of a Circle, came very near the Truth in all Sorts of Casks, which is thus perform'd :

1. By the Bung and Head Diameters, find such a mean Diameter as you judge will reduce the propos'd Cask to a Cylinder, by the Method laid down in *Page 453*. And then find its full Content, as in those *Examples*.

2. From the Bung Diameter subtract the mean Diameter and half the Difference, (*viz.* divide it by 2.)

3. From the wet Inches of the propos'd Ullage, subtract the said half Difference, and call it x ; then observe this Proportion.

Viz. { As the mean Diameter : is to 100 (the Diameter of the tabular Circle) :: so is the last Difference (*viz.* x) : to a versed Sine in the Table. (*Page 441.*)

Then if the tabular Segment, which stands against that versed Sine, be multiply'd into the Content of the Cask, the Product will shew the Ullage, *viz.* what Quantity of Liquor is either in the Cask, or drawn forth.

Example

Example 1. Let the Cask be that of the second Sort, in *Page 453. viz.* whose Bung Diameter is 31,5 Inches, mean Diameter 29,05, and the Content 98,71 Ale Gallons; and suppose there were 10,5 Inches wet in it, it is required to find the wet and dry Gallons?

Here $31,5 - 29,05 = 2,45$; its half is 1,22. And $10,5 - 1,22 = 9,28$
 Then $29,05 : 100 :: 9,28 : 0,319 = V.$ Sine; its Segm. is 0,2748
 And $98,71 \times 0,2748 = 27,12$ the Number of wet Gallons.

Again $31,5 - 10,5 = 21$ the dry Inches; and $21 - 1,22 = 19,78$
 Then $29,05 : 100 :: 19,78 : 0,68$; its Segment is 0,7241
 And $98,71 \times 0,7241 = 71,48$ the Number of dry Gallons.
 Proof $71,48 + 27,12 = 98,6$ the Contents of the Cask very near;
 which plainly shews the Truth of this Method.

Thus far may suffice concerning Gauging of Backs or Coolers, Tuns, Coppers, and Casks, &c. To which I shall only add, that as the Contents of all Brewers Utensils are to be computed by the Ale Gallons, so the Contents of all Distillers Utensils (*viz.* all their Wash-Backs, Stills, and Casks, &c.) must be computed by the Wine Gallon.

And in gauging of Malt (upon which there is now a Duty of four Shillings per Bushel) you must observe, That a Corn or Malt Bushel doth contain 2150,42 cubick Inches; (See *Page 42.*) and therefore in gauging of Malt-Cisterns, or other Vessels, 2150,42 will be a constant or fixed Divisor for finding the Area's of right-lin'd Figures in Bushels at one Inch deep, and 2738 will be a constant or fix'd Divisor for finding the Area's of circular Figures.

I have omitted the Business of gauging Mash-Tuns, and taking an Account of the Goods or Grains, in order to estimate what Quantity of Worts were produc'd from them, &c. because I could never find (by all my Observations) any Certainty therein; nor is it possible there should be any, by Reason of the great Difference that is in Malt (and its Grinding too) for the best Malt (well ground) will yield or produce the most Worts, and least Grains; on the contrary, bad Malt (being ill ground) yields the least Worts and most Grains.

A
SUPPLEMENT

Not in any of the former EDITIONS of this

BOOK.

Containing the

HISTORY

OF

LOGARITHMS,

WITH

Several easy METHODS of Constructing the *Tables* of
the LOGARITHMS and SINES, &c. Also the Demon-
stration of the AXIOMS and Doctrine of *Plane*

TRIGONOMETRY.

Extracted from the

Philosophical TRANSACTIONS and the WORKS of
Dr. KEIL, RONARNE, WARD, &c.

*Cuncta Trigonus habet, satagit quæ docta Mathesis,
Ille aperit clausum quicquid Olympus habet.*

T H E

P R E F A C E.

THE Mathematicks formerly received considerable Advantages; first, by the Introduction of the Indian Characters, and afterwards by the Invention of Decimal Fractions; yet has it since reaped at least as much from the Invention of Logarithms, as from both the other two. The Use of these, every one knows, is of the greatest Extent, and runs through all Parts of Mathematicks. By their Means it is that Numbers almost infinite, and such as are otherwise impracticable, are managed with Ease and Expedition. By their Assistance the Mariner steers his Vessel, the Geometrician investigates the Nature of the higher Curves, the Astronomer determines the Places of the Stars, the Philosopher accounts for other Phenomena of Nature; and lastly, the Usurer computes the Interest of his Money.

The Subject of the following Treatise has been cultivated by Mathematicians of the first Rank; some of whom, taking in the whole Doctrine, have indeed wrote learnedly, but scarcely intelligible to any but Masters. Others, again, accommodating themselves to the Apprehension of Novices, have selected out some of the most easy and obvious Properties of Logarithms, but have left their Nature and more intimate Properties untouched. My Design therefore, in the following Tract, is to supply what seemed still wanting, viz. to discover and explain the Doctrine of Logarithms, to those who are not yet got beyond the Elements of Algebra and Geometry.

The

The wonderful Invention of Logarithms we owe to the Lord Neper, who was the first that constructed and published a Canon thereof, at Edinburgh, in the Year 1614. This was very graciously received by all Mathematicians, who were immediately sensible of the extreme Usefulness thereof. And tho' it is usual to have various Nations contending for the Glory of any notable Invention, yet Neper is universally allow'd the Inventor of Logarithms, and enjoys the whole Honour thereof without any Rival.

The same Lord Neper afterwards invented another and more commodious Form of Logarithms, which he afterwards communicated to Mr. Henry Briggs, Professor of Geometry at Oxford, who was hereby introduced as a Sharer in the completing thereof: But, the Lord Neper dying, the whole Business remaining was devolved upon Mr. Briggs, who, with prodigious Application, and an uncommon Dexterity, compass'd a Logarithmic Canon, agreeable to that new Form for the first twenty Chiliads of Numbers (or from 1 to 20000) and for eleven other Chiliads, viz. from 90000 to 101000. For all which Numbers he calculated the Logarithms to fourteen Places of Figures. This Canon was publish'd at London in the Year 1624.

Adrian Vlacq published again this Canon at Goudæ in Holland in the Year 1628, with the intermediate Chiliads before omitted, filled up according to Briggs's Prescriptions; but these Tables are not so useful as Briggs's, because the Logarithms are continued but to 10 Places of Figures.

Mr. Briggs also has calculated the Logarithms of the Sines and Tangents of every Degree, and the hundredth Parts of Degrees to 15 Places of Figures, and has subjoined to them the natural Sines, Tangents, and Secants,

to 15 Places of Figures. The Logarithms of the Sines and Tangents are called *Artificial Sines and Tangents*, but the Sines and Tangents themselves are called *natural*. These Tables, together with their Construction and Use, were publish'd after Briggs's Death, at London, in the Year 1633, by Henry Gellibrand, and by him called *Trigonometria Britannica*.

Since then, there have been published, in several Places, compendious Tables, wherein the Sines and Tangents, and their Logarithms, consist of but seven Places of Figures, and wherein are only the Logarithms of the Numbers from 1 to 100000, which may be sufficient for most Uses.

The best Disposition of these Tables, in my Opinion, is that, first thought of by Nathaniel Roe, of Suffolk; and, with some Alterations for the better, followed by Sherwin in his *Mathematical Tables* published at London in 1705; wherein are the Logarithms from 1 to 101000 consisting of 7 Places of Figures. To which are subjoined the Differences and proportional Parts, by Means of which may be found easily the Logarithms of Numbers to 10000000, observing at the same Time that these Logarithms consist only of 7 Places of Figures. Here are also the Sines, Tangents, and Secants, with the Logarithms and Differences for every Degree and Minute of the Quadrant, with some other Tables of Use in practical Mathematicks.

T H E

THE CONSTRUCTION OF LOGARITHMS.

THESE most excellent and useful Numbers were first invented by the famous and never to be forgotten Lord *Neper*, Baron of *Merchiston* in *Scotland*, (afore said) *Ann.* 1614.) who ingeniously contriv'd to perform Multiplication and Division of Natural Numbers, by only adding or subtracting certain Artificial Numbers, which he called *Logarithms*, and the Extraction of Roots by dividing the Log. by 2 for the Square : by 3 for the Cube : by 4 for the Biquadrate, &c.

This Invention of his (no doubt) proceeded from a mature Consideration of the Coherence that is betwixt Numbers in Geometrical Proportion and those in Arithmetical Progression.

As in these following :

Viz. { 1 . 2 . 4 . 8 . 16 . 32 . 64 . 128, &c. Geometrical.
 { 0 . 1 . 2 . 3 . 4 . 5 . 6 . 7, &c. Arithmetical.

It is very perceptible, that, as the Numbers in the Geometrical Proportionals are produced by *Multiplication* or *Division*, those in the Arithmetical Progression are produced by *Addition* or *Subtraction* : As doth appear in this Example :

Viz. { $4 \times 32 = 128$ } or { $128 \div 32 = 4$ Geometr.
 { $2 + 5 = 7$ } { $7 - 5 = 2$ Arithmet.

Again, { 1 . 10 . 100 . 1000 . 10000 . 100000, &c. Geometr.
 { 0 . 1 . 2 . 3 . 4 . 5, &c. Arithmet.

The same Coherence is betwixt these latter, as was between the two first Ranks.

Viz. { $1000 \times 10 = 10000$ } or { $100000 \div 1000 = 100$ Geometr.
 { $3 + 1 = 4$ } { $5 - 3 = 2$ Arithmet.

Either of these Examples do sufficiently shew the Reason and very Ground of Logarithms.

And from the latter of these it was, that the prime Logarithms or Characters were first assigned.

As

As in this Table:

<i>Natural Num.</i>	<i>Logarithms.</i>
1	0,0000000
10	1,0000000
100	2,0000000
1000	3,0000000
10000	4,0000000
100000	5,0000000

Having laid this Foundation, the next Work was to find out the Logarithms of the intermediate Numbers situated betwixt 1 and 10, viz. of 2, 3, 4, 5, 6, 7, &c. and of those betwixt 10 and 100, viz. of 11, 12, 13, 14, 15, &c. and so on for the rest. This was a Work of some Difficulty, and very laborious.

The first Step in order thereunto (as I conceive) was to find out a Rank of continual Means betwixt 10 and 1, so as that the last (and least thereof) might be a mixed Number less than 2, and so near 1, as to have such a Number of Cyphers before the significant Figures thereof, as was intended the Places of Logarithms in the Table should consist of. Which Means are to be found, by extracting the square Root of 10 (having first annexed a competent Number of Cyphers thereunto;) then extracting the Root of that Root, and so by a continued Extraction of Root out of Root, until there be a Root so qualify'd as before-mention'd: Which, to make a Table to seven Places in the Logarithm, will require twenty-five several Extractions, the last of which will produce this Number, 1,00000006862238.

The next Step was to find out a Number betwixt (1) and (0) in Arithmetical Progression, that might truly correspond with the Mean before found (betwixt 10 and 1) such a Number must consequently be its Logarithm. And this may be found by a continual bisection (or halving) of 1, so often as was the Number of the foregoing Extractions (to wit, twenty-five) the last of which Bisections will produce 0,00000002980232, &c. the true Logarithm of 1,00000006862238.

For as 1,00000006862238 by twenty-five continued Involutions (viz. first into itself, then that Product into itself, and so on successively) will produce 10; so will 0,00000002980232, by the like Number of Doublings and Redoublings, produce 1.

This Mean (or Number) and its Logarithm being thus found, it will follow by Proportion, *As the significant Figures of this Mean : are to the significant Figures of its Logarithm : : so are the significant*

ficant Figures of any Mean, betwixt any given Number and 1 : (having seven Cyphers before such Figures, as this hath) to the significant Figures of its Logarithm. To which must be prefixed seven Cyphers to complete it. After which, being doubled, and redoubled according to the Number of Extractions required to produce its corresponding Mean, will at last discover the true Logarithm of the given Number. For the clearing of this, take an Example.

Suppose it were required to find the Logarithm of the Number 2, to seven Places. First, by a continued Extraction of Root out of Root, beginning at 2, find such a Mean, or Root as before, betwixt 2 and 1, as will have seven Cyphers before its significant Figures; which, after twenty-three several Extractions, will be this Number 1,00000008262958. Then, according to the foregoing Proportions, it will be $6862238 : 2980232 :: 8262958 : 3588557$. To which prefix seven Cyphers, as before directed, then will 1,00000008269958 have for its Logarithm, 00000003588557; which being doubled and redoubled, as abovesaid, will produce 0,30102997958658, the true Logarithm of 2; which being contracted to seven Places, according to the first Design, and agreeable to the seven Places of Cyphers, then it will become 0,3010299: But, in all the Tables that I have seen, the Logarithm of 2 is 0,3010300: I conceive the Reason is, because the remaining Figures 7958658 come so near Unity of the last Place in the retained Figures.

And, by the same Method that this Logarithm of 2 is made, may the Logarithm of any other Number be found. But when once the Logarithms of a few of the prime Numbers, viz. of 3. 7. 11. 13. &c. (that is, of such Numbers as cannot be produced by the multiplying of two Integer Factors) are obtained, the rest may be easily composed by Addition and Subtraction only. For as $3 \times 2 = 6$ so Log. of 3 + Log. of 2 = Log. of 6. And as $10 \div 2 = 5$ so Log. of 10 — Log. of 2 = Log. of 5. The like of all Numbers that have aliquot Parts (that is, such Integer Numbers as may be divided by Integers.) And indeed the Logarithms of several of the prime Numbers may also be obtained by Addition or Subtraction, as might easily be shewed, and is not difficult to conceive by any one, who but duly considers the Nature and Design of Logarithms, &c. of which I shall forbear saying any thing in this Place, and keep to my first Design herein, which was to give a brief Account of the ingenious Author's Method, as I conceive it, of making the same: who undoubtedly found it a very difficult Work, by Reason there are required so many several Extractions of Roots out of Roots, which must needs render it both troublesome and laborious. Then to propose a different Method of raising the Logarithms

rithms of such prime Numbers before-mentioned, which require the Extraction of Roots to obtain their respective Means, with one tenth Part of the Trouble and Time required by the foregoing Method. And not only so, but more exact; for, by our present Method of converging Series, the Root of any Power, how high soever it be, is easily found at one single Extraction: and thereby the Errors which would arise by extracting a *Surd Root* out of a *Surd Root*, especially when often repeated, are avoided; and consequently such a Mean, as may be required betwixt any Number and Unity, is thereby more exactly found.

Now, how this may be performed, I here intend to shew, as briefly as I can. In order thereunto, take this as a Model.

Let a = the Root, or Mean required betwixt any Number and Unity:

$$\text{Then } \begin{cases} a^2 = \square a & . a^4 = \square a^2 & . a^8 = \square a^4 \\ a^{16} = \square a^8 & . a^{32} = \square a^{16} & . a^{64} = \square a^{32} \\ a^{128} = \square a^{64} & . a^{256} = \square a^{128} & . a^{512} = \square a^{256} \end{cases}$$

And so on successively with the Indices in Geometrical Progression, until the Power of a be made equal to such a Term in that Progression, as that the Root, or Value of a may have, betwixt Unity and its significant Figures, so many Cyphers, as are the intended Number of Places in the Logarithms.

For Instance, let it be required to find the Mean between 10 and 1, then the Power of a must be $a^{33554432} = 10$, this Index 33554432 being the 25th Term in Geometrical Progression, which may be thus determined.

Let 1, the Characteristic or Logarithm of 10, be divided by such a Term in Geometrical Progression, as will cause such a Number of Cyphers to be before the significant Figures in the Quotient, as are required to be before the Figures of the Root a ; suppose 7, as before. Then $1 \div 33554432 = .00000002980232$, &c. which is the true Arithmetical Mean (as before found, by a continual bisecting of 1) correspondent to that signify'd by a ; and therefore the Value of a found by extracting the respective Root of $10 = a^{33554432}$ will be the Mean required; viz. 1,00000006862238 whose Log. is .00000002980232. These, being found, are the Foundation of the rest, as before.

Then suppose it be required to find the Logarithm of any of the prime Numbers; if you please, that of two. In order thereunto, let a = the Root or Mean sought betwixt 2 and 1, as before; then must a be continually involved, as by the above Model, until its Index be equal to the greatest Term in Geometrical Progression, whose Number of Places of Figures are to be equal to the Number of required Cyphers before a , to wit 7. According to which, the

Power

Power of a will be $a^{8388608} = 2$ (this 8388608 being the 23d Term in Geometrical Progression) consequently the respective Root of $2 = a^{8388608}$ will be the Mean requir'd.

Example.

Let $r + e = a$

Then will $r^{8388608} + 8388608 r^{8388607} e$
 $+ 35184367894528 r^{8388606} ee = a^{8388608} = 2$

Suppose $r = 1$

Then $1 + 8388608e + 35184367894528ee = 2$

That is $8388608e + 35184367894528ee = 1$

Each Part being divided by the Co-efficient found prefixed to ee , viz. 351843, &c. then it will become

$$,00000023e + ee = ,00000000000000284 = D$$

$$\text{Consequently } \left\{ \frac{D}{,00000023 + e} = e \right.$$

$$,00000000000000284 = D$$

$$+ e = ,000000008 \quad \begin{array}{r} 248 \\ \hline 36 \end{array} \quad (,00000008 = e$$

Divisor ,00000031

First $r = 1,$

$$+ e = ,00000008$$

New $r = 1,00000008$

which being duly involved, in the same Order as the Model denotes, and multiplied into the respective Co-efficients, will then produce these Numbers,

Viz. $1,9563638967 + 16411168e + 68833416066289ee = 2$

Then $16411168e + 68833416066289ee = ,0436361033$

And, $,0000002384e + ee = ,00000000000000063393 = D$

$$\text{Consequently } \left\{ \frac{D}{,0000002384 + e} = e \right.$$

$$,00000000000000063393 = D$$

$$4e = ,0000000026 \quad \begin{array}{r} 480 \\ \hline 15393 \end{array} \quad (,00000000263 = e$$

Divisor ,000000240

Divisor ,0000002410

000

Left

Mean. The next Thing will be to find the Logarithm of the Number from whence such Mean was produced, which may be thus performed :

First, find its corresponding Arithmetical Mean, or Logarithm, By Proportion (as in *Page 462.*) Then multiply that corresponding Mean, so found, into the Index Number of such Power as the Geometrical Mean was produced from; that Product will be the Logarithm of the given Number (without a continued Doubling and Redoubling, as before.) For the clearing of this, let it be required to complete the Logarithm of 2.

Having first found 1,00000006862238, the proper Geometrical Mean betwixt 10 and 1; also its corresponding Logarithm ,00000002980232 (as before directed) with them and the Mean betwixt 2 and 1, last found, viz. 1,0000000826295879; make use of the above-mentioned Proportion (as in *Page 463.*) viz.

$$6862238 : 2980232 :: 826295879 : 358855729$$

To which prefix seven Cyphers to complete it (as before.) Then it will become ,0000000358855729. This Number being multiplied into the Power of *a* (what that is, see *Page 465.*) will produce the Logarithm of 2.

$$\text{viz. } 0000000358855729 \times 8388608 = 0,30103000391352$$

But according to the first Design, it is required to have but seven Places, viz. 0301300; which is the true Logarithm of 2 without any Defect.

Thus I have presented you with a new and expeditious Method of making Logarithms; which if they were required to fourteen or fifteen Places (I can modestly say) they might then be made with one twentieth Part of the Time and Trouble required by the first Method.

M E T H O D III.

*A New Table of Logarithms. Compos'd by Mr. LONG.
Finding the Logarithm by Division only, and the Natural Number belonging to a Logarithm, by Multiplication only.*

Log.	Nat. Num.	Log.	Nat. Num.
0,9	7.943282347	0,00009	1.000207254
0,8	6.309573445	0,00008	1.000184224
0,7	5.011872336	0,00007	1.000161194
0,6	3.981071706	0,00006	1.000138165
0,5	3.162277660	0,00005	1.000115136
0,4	2.511886432	0,00004	1.000092106
0,3	1.995262315	0,00003	1.000069080
0,2	1.584893193	0,00002	1.000046053
0,1	1.258925412	0,00001	1.000023026
0,09	1.230268771	0,000009	1.000020724
0,08	1.202264435	0,000008	1.000018421
0,07	1.174897555	0,000007	1.000016118
0,06	1.148153621	0,000006	1.000013816
0,05	1.122018454	0,000005	1.000011513
0,04	1.096478196	0,000004	1.000009210
0,03	1.071519305	0,000003	1.000006908
0,02	1.047128548	0,000002	1.000004605
0,01	1.023292992	0,000001	1.000002302
0,009	1.020939484	0,0000009	1.000002072
0,008	1.018591388	0,0000008	1.000001842
0,007	1.016248694	0,0000007	1.000001611
0,006	1.013911386	0,0000006	1.000001381
0,005	1.011579454	0,0000005	1.000001151
0,004	1.009252886	0,0000004	1.000000921
0,003	1.006931669	0,0000003	1.000000690
0,002	1.004615794	0,0000002	1.000000460
0,001	1.002305238	0,0000001	1.000000230
0,0009	1.002074475	0,00000009	1.000000207
0,0008	1.001843766	0,00000008	1.000000184
0,0007	1.001613109	0,00000007	1.000000161
0,0006	1.001382506	0,00000006	1.000000138
0,0005	1.001151956	0,00000005	1.000000115
0,0004	1.000921459	0,00000004	1.000000092
0,0003	1.000691015	0,00000003	1.000000069
0,0002	1.000460623	0,00000002	1.000000046
0,0001	1.000230285	0,00000001	1.000000023

This

This Table I sometimes make use of for finding the Logarithm of any Number propos'd, and *vice versa*. Suppose I had Occasion to find the Logarithm of 2000. I look in the first Class of my Table (the whole Table consists of 8 Classes) for the next less to 2, which is 1.995262315, and against it is 3, which consequently is the first Figure of the Logarithm sought. Again, dividing the Number propos'd 2, by 1.995262315 the Number found in the Table, the Quotient is 1.002374467; which being look'd for in the second Class of the Table, and finding neither its Equal, nor a Lesser, I add 0 to the Part of the Logarithm before found, and look for the said Quotient, 1.002374467 in the third Class, where the next less is 1.002305238, and against it is 1, to be added to the Part of the Logarithm already found; and dividing the Quotient 1.002374467, by 1.002305238, last found in the Table, the Quotient is 1.000069070; which being sought in the fourth Class gives 0, but being sought in the fifth Class gives 2, to be added to the Part of the Logarithm already found; and dividing the last Quotient by the Number last found in the Table, *viz.* 1.000046053, the Quotient is 1.000023015, which, being sought in the sixth Class, gives 9 to the Part of the Logarithm already found: And dividing the last Quotient by the new Divisor, *viz.* 1.000002072, the Quotient is 1.000000219, which being greater than 1.000000115 shews that the Logarithm already found, *viz.* 3.3010299 is less than the Truth by more than half an Unit; wherefore adding 1, you have Briggs's Logarithm of 2000, *viz.* 3.3010300.

If any Logarithm be given, suppose 3,3010300, throw away the Characteristic, then overagainst these Figures 3 . . . 0 . . . 1 . . . 0 . . . 0, you have in their respective Classes 1,995262315 0 1,002305238 0 1,000069080 0 0 which multiply'd continually into one another, the Product is 2.000000019966, which, by reason the Characteristic is 3, becomes 2,000,000019966, &c. that is, 2000, the Natural Number desired. I shall not mention the Method by which this Table is fram'd, because you will easily see that from the Use of it.

It is obvious to the intelligent Reader, that these Classes of Numbers are no other than so many Scales of mean Proportionals: in the first Class, between 1 and 10; so that the last Number thereof, *viz.* 1,258925412 is the tenth Root of 10, and the rest in order ascending are the Powers thereof. So in the second Class, the last Number 1,023292992 is the hundredth Root of 10, and the rest in the same Manner are Powers thereof. So 1.002305238, in the third Class, is the tenth Root of the last of the second, and
the

the rest its Powers, &c. Or, which is all one, each Number, in the preceding Class, is the tenth Power of the corresponding Number in the next following Class: Whence 'tis plain, that to construct these Tables requires no more than one Extraction of the fifth or sursolid Root for each Class, the rest of the Work being done by the common Rules of Arithmetick.

METHOD IV.

Their Construction, according to the common Rules, given by many Extractions of Roots, is tedious; the best Way yet known is this which follows.

To make a Table of Logarithms.

First, Put for the Logarithm of 1 a Cypher for the Index, and a competent Number of Cyphers for the Logarithm, according to the Number of Places you would have your Logarithms consist of; for 10 and Unit, with the same Number of Cyphers; for 100, 2, with as many Cyphers; for 1000, 3, with as many Cyphers, &c.

Secondly, Find the Difference between some two Logarithms above 1000, or rather above 10000, that differ by Unity; thus multiply the two Numbers together, and that Product you must multiply again by 43429448190325183896 * which last Product divided by the Arithmetical Mean between both Numbers, the Quotient is the Difference sought.

Suppose we would find the Difference between the Log. 10000, and 10001, the Product of these two Numbers is 1.00010000. which multiplied by 4343 produced 43434343; this divided by 10000.5, quotes 4343. Now if to the Logarithm of 10000, which is 4.00000000, you add the Difference before found, to wit, 434, the Sum 4.0000434 is the true Logarithm of 10001 to 7 Places.

Thirdly, Having thus found the Difference of any two Logarithms differing by Unity, and consequently some of the Logarithms by dividing the Difference found by the Arithmetical Mean, between any two Numbers differing by Unity, you shall have the Difference of the Logarithm of those two Numbers.

Thus to find the Difference betwixt the Logarithm of 274, and 275; divide 4343, the Difference of the Logarithm of 10000, and 10001 by 2745 the Quotient 15821, is the Difference sought.

Fourthly, Having by this Means found a few of the prime Logarithms, the rest are made by Addition and Subtraction, and hav-

ing

* Which is the Subtangent of the Curve expressing Briggs's Logarithms. See Keil's Trig. Pag. 135, 140, &c.

ing made the Canon upwards, above 1000 to 10000, by Consequence it is made for all inferior Numbers.

The prime Numbers to which Logarithms must be found, in the first Place are these, 2 . 3 . 7 . 11 . 13 . 17 . 19 . 23 . 29 . 31 . 37 . 41 . 43 . 47 . 53 . 59 . 61 . 67 . 71 . 73 . 79 . 89 . 97, &c. or the same Numbers with Cyphers.

But since it was very tedious and laborious, to find the Logarithms of the prime Numbers, and not easy to compute Logarithms by Interpolation, by first, second, and third, &c. Differences, therefore the great Men, Sir *Isaac Newton*, *Mercator*, *Gregory*, *Wallis*, and lastly, Dr. *Halley*, have published infinite converging Series, by which the Logarithms of Numbers to any Number of Places may be had more expeditiously and truer: Concerning which Series, Dr. *Halley* has written a learned Tract, in the *Philosophical Transactions*, wherein he has demonstrated those Series after a new Way, and shews how to compute the Logarithms by them. But I think it may be more proper here to add a new Series, by Means of which may be found, easily and expeditiously, the Logarithms of large Numbers.

Let z be an odd Number, whose Logarithm is sought; then shall the Number $z - 1$ and $z + 1$ be even, and accordingly their Logarithms, and the Difference of the Logarithms will be had, which let be called y : Therefore, also the Logarithm of a Number which is a Geometrical Mean between $z - 1$ and $z + 1$ will be given, viz. equal to the half Sum of the Logarithms. Now the Series $y \times \frac{1}{4z} + \frac{1}{24z^3} + \frac{7}{360z^5} + \frac{181}{15120z^7} + \frac{13}{25200z^9}$ &c. shall be equal to the Logarithm of the Ratio, which the Geometrical Mean between the Numbers $z - 1$ and $z + 1$, has to the Arithmetical Mean, viz. to the Number z .

If the Number exceeds 1000, the first Term of the Series $\frac{y}{4z}$ is sufficient for producing the Logarithm to 13 or 14 Places of Figures, and the second Term will give the Logarithm to 20 Places of Figures. But if z be greater than 10000, the first Term will exhibit the Logarithm to 18 Places of Figures; and so this Series is of great Use in filling up the Logarithms of the Chiliads omitted by *Briggs*. For Example; It is required to find the Logarithm of 20001. The Logarithm of 20000 is the same as the Logarithm of 2, with the Index 4 prefix'd to it; and the Difference of the Logarithms of 20000 and 20002, is the same as the Difference of the Logarithms of the Numbers 10000 and 10001, viz. 0.00004342725687. And if this Difference be divided by $4z$, or 80004, the Quo-

Quotient $\frac{y}{4x}$ shall be _____ 0. 00000 0000542814

And if the Logarithm or the Geometrical Mean be added to the Quotient, the _____ 4. 30105 1709302416

Sum will be the Logarithm of 20001. 4. 30105 1709845230

Wherefore it is manifest, that to have the Logarithm to 14 Places of Figures, there is no Necessity of continuing out the Quotient beyond 6 Places of Figures. But if you have a Mind to have the Logarithm to 10 Places of Figures only, as they are in *Vlacq's* Tables, the two first Figures of the Quotient are enough. And if the Logarithms of the Numbers above 20000 are to be found by this Way, the Labour of doing them will mostly consist in setting down the Numbers. *Note*, This Series is easily deduced from that found out by *Dr. Halley*; and those who have a Mind to be inform'd more in this Matter, let them consult his abovenam'd Treatise.

Mr. *WARD's* Easy Method of making the Canon of Sines, Tangents, &c.

FIRST, let me premise two Things, that the Periphery of a Circle, whose Radius is Unity or 1, is 6.283185, &c. and that the natural Sine of one Minute doth so insensibly differ from the Length of the Arch of one Minute, that it may be taken for the same.

Consequently, $\left\{ \begin{array}{l} \text{As the Periphery in Minutes : is to the Periphery in equal Parts of the Radius : : so is} \\ \text{one Minute : to the Parts agreeing to that Minute.} \end{array} \right.$

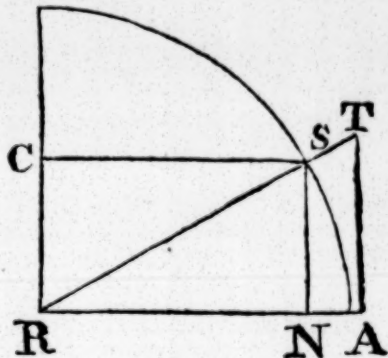
That is, 21600' : 6,283185 : : 1' : 0,000290888 = the Natural Sine of one Minute; which agrees with the largest Table of Sines I ever saw.

Having thus got the Sine of one Minute, its Co-sine may be thus found :

Suppose

Suppose $RA = RS$ the Radius of any Circle, $SN =$ the Sine of the Arch SA . Then $RN = CS$ is the Co-Sine of that Arch. But $\square RS = \square SN + \square RN$, consequently $\sqrt{\square RS - \square SN} = RN$.

That is, From the Square of the Radius, subtract the Square of the Sine of $1'$, the square Root of the Remainder will be the Co-Sine of $1'$, per *Chap. 9. Prop. 1.* In Numbers, the Sine of $1'$ is 000290885, its Square is 0,000000084612; and $1 - 0,000000084612 = 0,999999915388$, the Square Root thereof is ,99999995 = the Co-Sine required.



The Sine and Co-Sine of one Minute being thus obtain'd, all the rest of the Sines in the Quadrant may be gradually calculated by Mr. *Michael Dary's* Sinical Proportions; which I shall here insert, to the same Effect as they are in his *Miscellanies*; and then explain and demonstrate the Truth of those Proportions.

If a Rank of Arches be equi-different;

Then $\left\{ \begin{array}{l} \text{As the Sine of any Arch in that Rank : is to the Sum of the} \\ \text{Sines of any two Arches equally remote from it on each Side : :} \\ \text{so is the Sine of any other Arch in the said Rank : to the Sum} \\ \text{of the Sines of two Arches next it on each Side ; having the} \\ \text{same common Distance.} \end{array} \right.$

Immediately after these Proportions, he lays down the following *Æquations*:

Three Arches equi-different being proposed; if (saith he) you put $Z =$ the Sine of the great Extreme, $y =$ the Sine of the lesser Extreme; $M =$ the Sine of the Mean; $m =$ the Co-Sine thereof; D the Sine of the common Difference; $d =$ the Co-Sine thereof; and $R =$ the Radius.

$$1. \text{ Then } Z + y = \frac{2Md}{R}. \text{ Then } Z - y = \frac{2mD}{R}$$

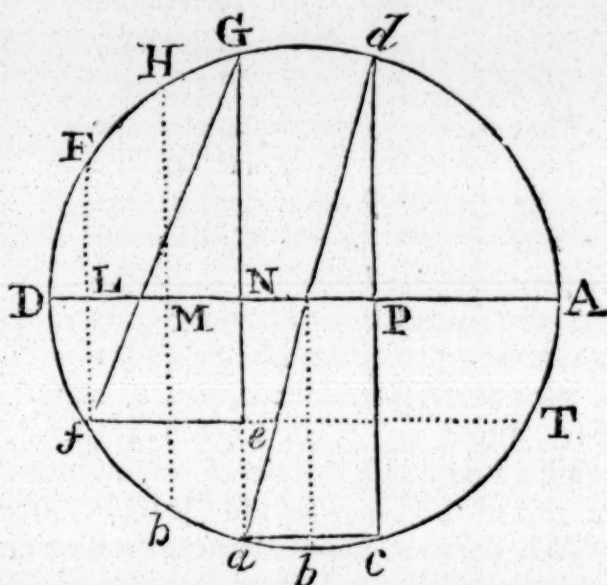
$$3. \text{ Then } Zy = MM - DD. \text{ 4. Then } \frac{Z}{y} = \frac{Md + mD}{Md - Md}$$

From the foregoing it is evident (saith he) that if two Thirds, viz. either the former or latter 60 Degrees, or the former 30 Degrees, and the latter 30 Degrees of the Quadrant be completed with Sines; the remaining Part of the Quadrant may be completed by Addition, or Subtraction only.

Thus far is from the ingenious Mr. Dary, concerning these excellent Proportions; the Truth whereof I shall thus demonstrate.

In the annexed Circle $DA = da$ are Diameters, $fb = ba = ab = bc$, are equal Arches.

Draw fT parallel to DA ; then will $Ne = Lf$. And the $\triangle dac$, like the $\triangle Gfe$, being both right-angled at c and e , and $\angle d = \angle G$ because subtended by the equal Arches $ac = fa$.



Therefore $da : dc :: Gf : Ge$.

Consequently $\frac{1}{2} da : dc :: \frac{1}{2} Gf : Ge$. But $Hb = Gf$, whence $HM = \frac{1}{2} Gf$, and $\frac{1}{2} da =$ the Radius, $dp = \frac{1}{2} dc$. Therefore it will be, Radius : $2 dp :: HM = \frac{1}{2} Gf : GN + Ne = GN + Lf$. That is, as the Radius : is to twice the Sine $dp ::$ so is the Sine HM : to the Sum of the two Sines GN and $FL = fL$. Q. E. D.

I shall now explain these Proportions, and shew how they may be applied in Practice: Having the Sine of one Minute, and its Co-Sine as before; let the Radius be made the mean or middle Term between those two Extremes; then the Proportions will run

Thus $\left\{ \begin{array}{l} \text{As the Radius : is to the double Co-sine of one Minute : : so is} \\ \text{the Sine of one Minute : to the Sine of two Minutes, and of} \\ \text{00' : and so is the Sine of 2' : : to the Sum of the Sines of 3'} \\ \text{and 1' : : and so is the Sine of 3' : to the Sum of the} \\ \text{Sine of 4' and 2'.} \end{array} \right.$

And so on in a successive Order, from Minute to Minute.

And then, if from the Sum of the Sines of $3'$ and $1'$ be taken the Sine of $1'$, the Remainder will be the Sine of $3'$: And the like, if, from the Sum of the Sines of $4'$ and $2'$, be taken the Sine of $2'$ there will remain the Sine of $4'$, &c.

Proceeding on by this Method, all the Natural Sines in the Quadrant may be easily calculated by Addition, and Subtraction only. For the Radius, or First Term in the Proportion, being 1,0000000

or Unity, Division is wholly avoided: And because the second Term in the Proportion varies not, if a Tariffa, or small Table be made thereof, to all the nine Digits, then Multiplication is also avoided. For, by the Help of that Tariffa, the whole Work may be perform'd by Addition and Subtraction, until all the Sines are gradually made.

Thus you have an easy Way of making the Canon of Sines; which being once done, the Tangents and Secants may be found by the following

Proportions $\left\{ \begin{array}{l} \text{As the Co-sine of any Arch: is to the Sine of that Arch:} \\ \text{so is the Radius: to the Tangent of the same Arch.} \end{array} \right.$

That is, by the first Scheme of this Problem,

$RN:SN::RA:TA$. And $RN:RS::RA:RT =$ the Secant of that Arch.

Plane Trigonometry.

DEFINITIONS.

1. A Circle is suppos'd to be divided into 360 equal Parts, called Degrees; and each Degree into 60 equal Parts, called Minutes; and each Minute into 60 equal Parts, called Seconds, &c. Any Portion of whose Circumference is called an Arch, and is measured by the Number of Degrees it contains.

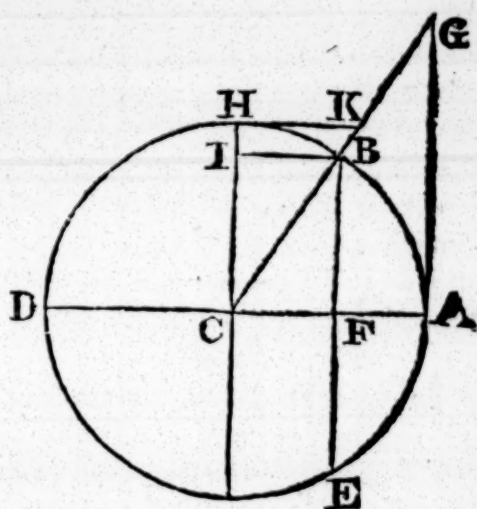
2. A Chord or Subtense is a straight Line, connecting the Extremities of an Arch; as BE is the Chord of the Arches BAE, BDE.

3. A Sine (or Right-sine) is a straight Line drawn from one End of an Arch perpendicular to that Diameter passing thro' the other End; or it is half the Chord of twice the Arch; so BF is the Sine of the Arches BA, BD. And here it is evident, that the Sine of 90 Degrees (which is equal to the Radius or Semi-Diameter of the Circle) is the greatest of all Sines, the Sine of an Arch greater than a Quadrant being less than the Radius.

4. The Difference of an Arch from a Quadrant, whether it be greater or less, is call'd its Complement; so HB is the Complement of the Arches BA, BD; BI is the Sine of that Complement,

P p p 2

and



and therefore it is called the Co-sine, or Sine-Complement of the Arches BA, BD.

5. The Secant of an Arch is a straight Line drawn from the Center thro' one End of the Arch till it meet with the Tangent, which is a straight Line touching the Circle at the Extremity of that Diameter which cuts the other End of the Arch; so CG is the Secant, and AG the Tangent of the Arches BA, BD: And CK is the Co-secant, and HK the Co-tangent of the said Arches.

6. A Versed Sine is the Segment of the Diameter intercepted between the Arch and its Sine: Thus FA is the Versed Sine of the Arch BA, and FD of the Arch BD.

7. Whatever Number of Degrees an Arch wants of a Semi-Circle is called its Supplement.

8. That Part of the Radius which is betwixt the Center and Sine is equal to the Co-sine; thus CF is $=$ IB.

9. If an Arch be greater or less than a Quadrant the Sum or Difference of the Radius or Co-sine is equal to the Versed Sine.

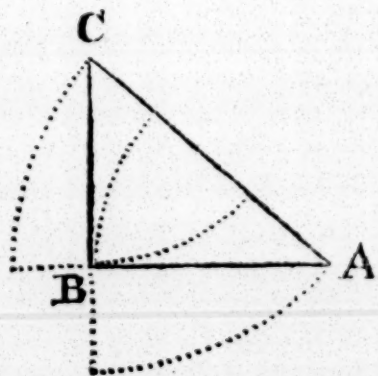
In a Triangle are six Parts, viz. three Sides and three Angles: Any three of which being given, except the three Angles of a Plain Triangle, the other three may be found either Mechanically, by the Help of a Scale of equal Parts and Line of Chords, or by an Arithmetical Calculation, if, supposing the Radius divided into any Number of equal Parts, we know how many of those equal Parts are in the Sine, Tangent, or Secant of any Arch propos'd: The Art of inferring which is called *Trigonometry*, and it is either Plane or Spherical.

Plane Trigonometry is solv'd by the Help of four fundamental Propositions call'd *Axioms*.

Axiom I.

In a Right-angled Triangle ABC, if one Leg of the Right-angle, as AB or CB, be made the Radius of a Circle, then shall the other Leg CB or AB be the Tangent of the Angle opposite to it, and the Hypothenuse AC (or Side opposite to the Right-angle) its Secant (by Definition 5.)

But if the Hypothenuse AC be made the Radius of a Circle, then will the Legs (or Sides including the Right-angle) to wit, CB and AB be the Sines of the Angles opposite (by Definition 3.)



Upon

Upon this *Axiom* depends the Solution of the seven Cases of Right-angled Plane Triangles.

Note, That the three Angles of a Plane Triangle make two Right-Angles, or 180 Degrees, by 32. 1 *Eucl.*

For the more easy making the Proportions for the Solution of Right-angled Triangles, observe, that as different Sides are made Radius, so the other Sides require different Names, which Names are either Sines, Tangents, or Secants, and are to be taken out of your Table.

To find a Side, any Side may be made Radius: Then say, as the Name of the Side given is to the Name of the Side required; so is the Side given to the Side required.

But to find an Angle, one of the given Sides must be made Radius; then, as the Side made Radius, is to the other Side; so is the Name of the first Side (which is Radius) to the Name of the second Side; which fourth Proportional must be found among the Sines or Tangents, &c. to be determined by the Side made Radius, and against it is the Angle required.

The Proportions for the Solution of seven Cases of Plane Right-angled Triangles.

[See the next foregoing Fig.]

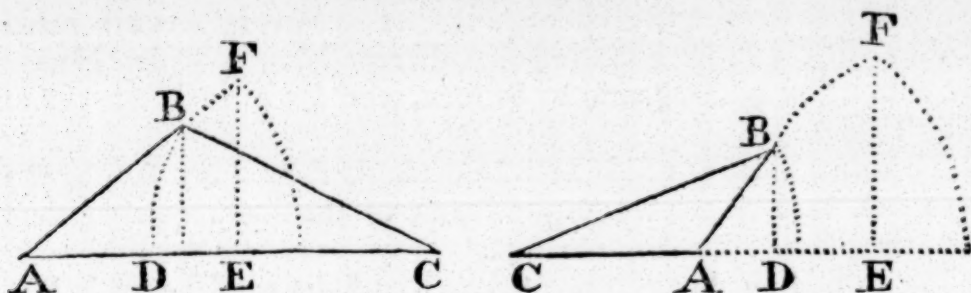
Given.	Reqd.	Proportions.	Rad.	Cafe.
AB A and C	BC	Cof. A : Si. A :: AB : BC. R : Tan. A :: AB : BC. Co-t. A : R :: AB : BC.	AC AB BC	1
AB A and C	AC	Cof. A : R :: AB : AC. R : Sec. A :: AB : AC. Tan. A : Cofe. A :: AB : AC.	AC AB BC	2
AB BC	A and C	AB : BC :: R : Tan. A. Complement is C. BC : AB :: R : Tan. C. Complement is A.	AB BC	3
AB BC	AC	AB : BC :: R : Tan. A; then Cof. A : R :: AB : AC, or $\sqrt{\square AB + \square BC} = AC$ (per 47. 1. <i>Eucl.</i>)	AB AC	4
AB AC	A and C	AC : BC :: R : Cof. A. AB : AC :: R : Secant A.	AC AB	5
AB AC	BC	AC : AB :: R : Cof. A; then R : Tan. A :: AB : BC, or $\sqrt{\square AC - \square AB} = BC$.	AC AB	6
AC A and C	AB	R : Cof. A :: AC : AB. Sec. A : R :: AC : AB. Cof. A : Cot. A :: AC : AB.	AC AB BC	7

Axiom

Axiom II.

In any Triangle the Sides are proportional to the Sines of the opposite Angles.

Demonstration.



Produce the lesser Side of the Triangle ABC, to wit AB to F, making $AF = BC$: Let fall the Perpendiculars BD, FE, upon the Side CA produc'd if Need be; then will FE be the Sine of the Angle A, and BD the Sine of the Angle C, to the Radius $BC = AF$.

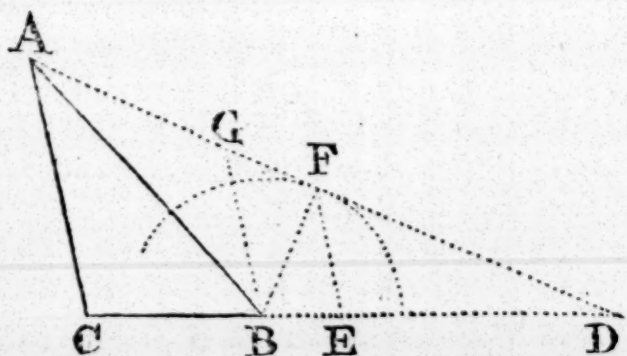
Now the Triangles ABD and AFE, having the $\angle A$ common to them both, and the $\angle D = \angle E =$ to a Right-Angle, are similar; wherefore (by 4. 6 *Eucl. Elem.*) $AF (BC) : AB :: FE : BD$; viz. $:: \text{Si. A} : \text{Si. C}$. *Q. E. D.* Otherwise thus; by *Ax. I.* $AB : R :: BD : \text{Si. A}$, and $BC : R :: BD : \text{Si. C}$; therefore $AB \times \text{Si. A} (= R \times BD) = BC \times \text{Si. C}$; wherefore $AB : BC :: \text{Si. C} : \text{Si. A}$. *Q. E. D.*

Axiom III.

The Sum of the Legs of any Angle of a Plane Triangle is to their Difference, as the Tangent of half the Sum of the Angles opposite to those Legs is to the Tangent of half their Difference.

Demonstration.

In the Triangle ABC produce CB, the lesser Leg of the Angle B, till BD becomes $= BA$, the greater Leg, and then bisect CD in E; join A D and bisect it also in F; draw BF, which (by 8. 1 *Eucl. El.*) will be perpen. to AD; and draw EF, which (by 2. 6 *Eucl. Elem.*) will be parallel to AC. Then will the Angle $ABF = FBD = \frac{1}{2} ABD$, which external Angle ABD is (by 32. 1 *Eucl. Elem.*) $= BAC + C$, that is to the Sum of the opposite Angles required.



Draw then BG parallel to AC, so will the Angle GBA be (by 29. 1 *Eucl. Elem.*) equal to its Alternate one BAC; and if from half the Sum

Sum of the opposite Angles you take the lesser Angle, *i. e.* If from $\angle ABF$ you take the $\angle GBA$, there will remain $\angle GBF =$ half the Difference of the opposite Angles : And so also, if from CE half the Sum of the Legs, you take CB the lesser Leg, there will remain BE equal to half the Difference of the Legs. And then since the $\triangle ABF$ is Right-angled, if BF be made Radius, AF will be the Tangent of $\angle ABF$ (*i. e.* the Tangent of half the Sum of the opposite Angles) ; and in the little $\triangle GBF$, FG will be the Tangent of the $\angle GBF$ (*i. e.* the Tangent of half the Difference of the opposite Angles) : But the Segments of the Legs of any Triangle cut by Lines parallel to the Base, being (by *Schol.* to 2. 6 *Eucl. El.*) proportional ; $EC : EB :: FA : FG$; that is in Words, half the Sum of the Legs is to half their Difference, as the Tangent of half the Sum of the opposite Angles is to the Tangent of half their Difference : But Wholes are as their Halves ; wherefore the Sum of the Legs is to their Difference, as the Tangent of half the Sum of the Angles opposite is to the Tangent of half their Difference. Q. E. D.

Axiom IV.

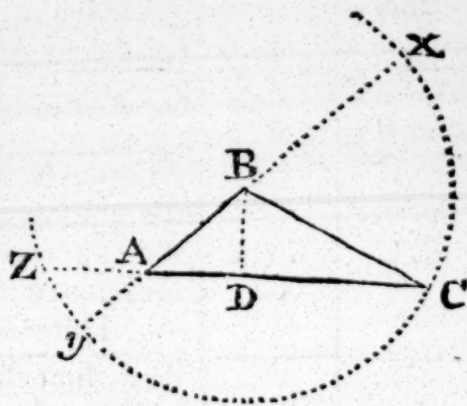
The Base or greatest Side of any Plane Triangle is to the Sum of the Legs, as the Difference of the Legs is to the Difference of the Segments of the Base made by a Perpendicular let fall from the Angle opposite to the Base.

Demonstration.

From the $\angle B$, on the Base AC , of the $\triangle ABC$, let fall the Perpendicular BD ; on B , as a Center, with the greater Leg BC , as a Radius, describe the Circle $Bx Cy Z$; and produce AB to x and y , and CA to Z . Then by the 35. 3 *Eucl. Elem.* $Ay \times Ax = AC \times AZ$; *viz.* : $BC - BA : x : BC + BA : = AC \times : DC - DA$: therefore $AC : BC + BA :: BC - BA : DC - DA$. Q. E. D. Otherwise, let the Difference of the Squares of the Sides BC and AB be taken and divided by the Base AC , the Quotient shall be the Difference of the Segments of the Base aforesaid : Or, square all the 3 Sides, and deduct the Square of one of the less Sides out of the Sum of the other two Squares, divide half the Remainder by the longest Side, the Quotient is the Alternate Segment of the Base.

The Proportions for the Solution of the six Cases of Plane oblique Triangles.

[See the last Fig.]



Given.

Given.	Reqd.	Proportions.	Ax.	Case.
AB BC and C	A	$AB : BC :: \text{Si. } C : \text{Si. } A.$	2	I

N. B. 1st, If the given Angle be Obtuse, the other two Angles then are each Acute.
 2^{dly}, If the Side opposite to the given Angle is longer than the Side opposite to the Angle sought, then is the Angle sought Acute; but if shorter, then is the said Angle doubtful, and may be either Acute or Obtuse, because both the Sine and its Complement to two Right-Angles are the same: Wherefore to be certain of what Quality the Angle opposite to the greatest Side is: Take the Sum and Difference of the greatest Side and Middle (or least) and their Logarithms, if the Half of them be equal to the Logarithm of the third Side, the Angle opposite to the greatest Side is a Right-Angle; but if the Logarithm of the third Side be greater than the Half, it is Acute, if less, it is Obtuse: Or, without Logarithms, multiply the said Sum by the Difference abovesaid, and extract the Square Root,

which if $\left\{ \begin{array}{l} \text{Equal to} \\ \text{Greater than} \\ \text{Less than} \end{array} \right\}$ the third Side, then is the greatest Angle $\left\{ \begin{array}{l} \text{Right} \\ \text{Obtuse} \\ \text{Acute} \end{array} \right\}$

AB BC and C	AC	$AB : BC :: \text{Si. } C : \text{Si. } A.$ Hence, by Subtraction, the $\angle B$ will be known. $\text{Si. } A : \text{Si. } B :: BC : AC.$	2	2
A, C and BC	AB	$\text{Si. } A : \text{Si. } C :: BC : AB.$	2	3
B AB BC	A and C	$BC + AB : BC - AB :: \text{Tan. } \frac{1}{2} \text{ Sum of the } \angle\text{s opposite} : \text{Tan. } \frac{1}{2} \text{ Difference of the } \angle\text{s opposite.}$ Then $\frac{1}{2} \text{ Sum} + \frac{1}{2} \text{ Difference} = \text{greater } \angle A$; and $\frac{1}{2} \text{ Sum} - \frac{1}{2} \text{ Difference} = \text{lesser } \angle C.$	3	4
B AB BC	AC	First, find the Angles by the last; then $\text{Si. } C : \text{Si. } B :: AB : AC.$	3 2	5
AB BC AC	A B C	$AC : BC + BA :: BC - BA : DC - DA:$ Then $\frac{1}{2} AC + \frac{1}{2} DC - \frac{1}{2} DA = DC.$ And $\frac{1}{2} AC - \frac{1}{2} DC - \frac{1}{2} DA = DA.$ Then $AB : AD :: R : \text{Cofi. } A.$ And $CB : DC :: R : \text{Cofi. } C.$ And $180^\circ - \angle A - \angle C = \angle B.$	4 1 1	6

Or more readily at one Operation.

From half the Sum of the Sides subduct each particular Side, and let the Sum of the Logarithm of the half Sum and Difference of the Side subtending the enquired Angle be subducted from the Sum of the Log. of the other Difference and the double Radius, half the Remainder shall be the Log. of the Tangent of half the enquired Angle.

Agreeable to this Axiom in Gellibrand's Trig. Britannica, p. 46.

As the Rectangle of half the Sum of the Sides and the Difference between that half Sum and the Side opposite to the Angle required, is to the Rectangle of the other two Remainders; so is the Square of Radius to the Square of the Tangent of half the Angle sought.

Ex Angulis latera, vel ex lateribus Angulos & mixtim in Triangulis tam planis quam Sphaericis assequi, Summa Gloria Mathematici est: Sic enim Caelum & Terras & Maria felici & samirando calculo mensurat.

Frans. Vieta:

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$$\sqrt{\frac{147}{3}} - \sqrt{\frac{32}{2}} = 5$$

$$⑥^2 \quad \frac{147}{3} - 2\sqrt{\frac{147}{3}} \times \sqrt{\frac{32}{2}} + \frac{32}{2} = 9$$

viz.

$$\left. \begin{array}{l} \frac{147}{3} = 49 \\ \text{and } \frac{32}{2} = 16 \end{array} \right\} = 50 - 4 \times 4 = 9$$

$$\frac{48}{16} - \frac{24}{3} = 1$$

$$\left. \begin{array}{l} ⑥^2 \quad \frac{2304}{16} - 2 \frac{48}{16} \times \frac{24}{3} + \frac{576}{9} \end{array} \right\} = 16$$

viz.

$$\left. \begin{array}{l} \frac{2304}{16} = 144 \\ \text{and } \frac{576}{9} = 64 \end{array} \right\} = 158 - 24 \times 8 = 16$$

2 y-b-r

X.m

John

